Excerpts From

Ayers, F. 1952, Theory and Problems of Differential Equations.

Spiegel, R. 1956, Theory and Problems of Vector Analysis and an Introduction to Tensor Analysis.

Chapter 1

VECTORS and SCALARS

A VECTOR is a quantity having both magnitude and direction, such as displacement, velocity, force, and acceleration.

Graphically a vector is represented by an arrow *OP* (Fig.1) defining the direction, the magnitude of the vector being indicated by the length of the arrow. The tail end *O* of the arrow is called the *origin* or *initial point* of the vector, and the head *P* is called the *terminal point* or *terminus*.

Analytically a vector is represented by a letter with an arrow over it, as $\vec{A}$ in Fig.1, and its magnitude is denoted by $|\vec{A}|$ or A. In printed works, bold faced type, such as **A**, is used to indicate the vector $\vec{A}$ while $|\mathbf{A}|$ or A indicates its magnitude. We shall use this bold faced notation in this book. The vector OP is also indicated as $\overrightarrow{OP}$ or **OP**; in such case we shall denote its magnitude by $\overline{OP}$, $|\overrightarrow{OP}|$, or $|\mathbf{OP}|$.

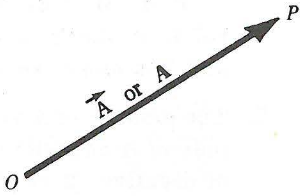


Fig.1

A SCALAR is a quantity having magnitude but no direction, e.g. mass, length, time, temperature, and any real number. Scalars are indicated by letters in ordinary type as in elementary algebra. Operations with scalars follow the same rules as in elementary algebra.

VECTOR ALGEBRA. The operations of addition, subtraction and multiplication familiar in the algebra of numbers or scalars are, with suitable definition, capable of extension to an algebra of vectors. The following definitions are fundamental.

1. Two vectors **A** and **B** are *equal* if they have the same magnitude and direction regardless of the position of their initial points. Thus **A** = **B** in Fig.2.
2. A vector having direction opposite to that of vector **A** but having the same magnitude is denoted by $-\mathbf{A}$ (Fig.3).

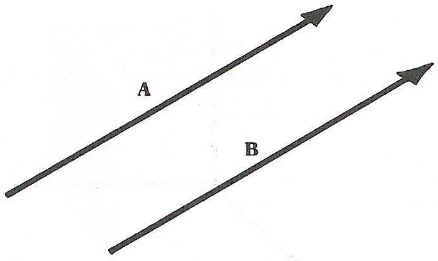


Fig. 2

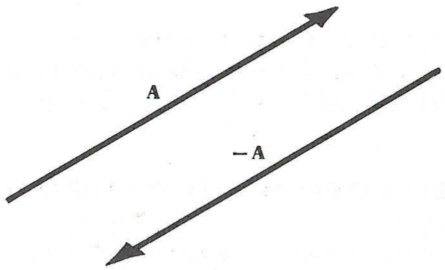


Fig. 3

3. The *sum* or *resultant* of vectors **A** and **B** is a vector **C** formed by placing the initial point of **B** on the terminal point of **A** and then joining the initial point of **A** to the terminal point of **B** (Fig.4). This sum is written **A**+**B**, i.e. **C** = **A**+**B**.

   The definition here is equivalent to the *parallelogram law* for vector addition (see Prob.3).

   Extensions to sums of more than two vectors are immediate (see Problem 4).

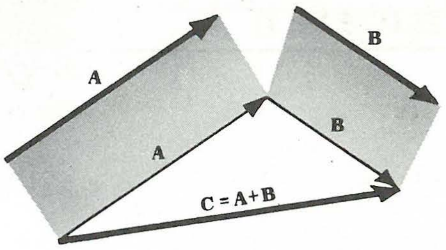


Fig. 4

4. The *difference* of vectors **A** and **B**, represented by **A**−**B**, is that vector **C** which added to **B** yields vector **A**. Equivalently, **A**−**B** can be defined as the sum **A**+(−**B**).

   If **A** = **B**, then **A**−**B** is defined as the *null* or *zero vector* and is represented by the symbol **0** or simply 0. It has zero magnitude and no specific direction. A vector which is not null is a *proper vector.* All vectors will be assumed proper unless otherwise stated.

5. The *product* of a vector **A** by a scalar m is a vector $m\mathbf{A}$ with magnitude $|m|$ times the magnitude of **A** and with direction the same as or opposite to that of **A**, according as m is positive or negative. If $m = 0$, $m\mathbf{A}$ is the null vector.

LAWS OF VECTOR ALGEBRA. If **A**, **B** and **C** are vectors and m and n are scalars, then

1. $\mathbf{A}+\mathbf{B} = \mathbf{B}+\mathbf{A}$	Commutative Law for Addition
2. $\mathbf{A}+(\mathbf{B}+\mathbf{C}) = (\mathbf{A}+\mathbf{B})+\mathbf{C}$	Associative Law for Addition
3. $m\mathbf{A} = \mathbf{A}m$	Commutative Law for Multiplication
4. $m(n\mathbf{A}) = (mn)\mathbf{A}$	Associative Law for Multiplication
5. $(m+n)\mathbf{A} = m\mathbf{A}+n\mathbf{A}$	Distributive Law
6. $m(\mathbf{A}+\mathbf{B}) = m\mathbf{A}+m\mathbf{B}$	Distributive Law

Note that in these laws only multiplication of a vector by one or more scalars is used. In Chapter 2, products of vectors are defined.

These laws enable us to treat vector equations in the same way as ordinary algebraic equations. For example, if **A**+**B** = **C** then by transposing **A** = **C**−**B**.

A UNIT VECTOR is a vector having unit magnitude, If **A** is a vector with magnitude $A \neq 0$, then **A**/A is a unit vector having the same direction as **A**.

Any vector **A** can be represented by a unit vector **a** in the direction of **A** multiplied by the magnitude of **A**. In symbols, **A** = A**a**.

THE RECTANGULAR UNIT VECTORS i, j, k. An important set of unit vectors are those having the directions of the positive x, y, and z axes of a three dimensional rectangular coordinate system, and are denoted respectively by **i**, **j**, and **k** (Fig.5).

We shall use *right-handed rectangular coordinate systems* unless otherwise stated. Such a system derives

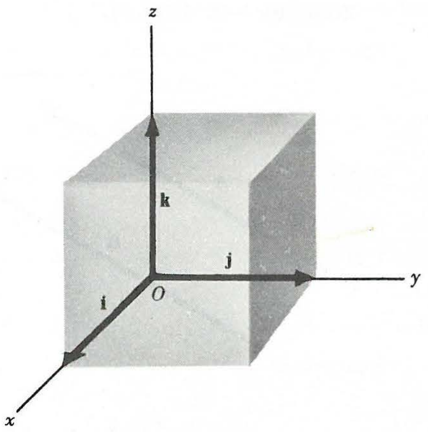


Fig. 5

its name from the fact that a right threaded screw rotated through 90° from Ox to Oy will advance in the positive z direction, as in Fig.5 above.

In general, three vectors **A**, **B** and **C** which have coincident initial points and are not *coplanar*, i.e. do not lie in or are not parallel to the same plane, are said to form a *right-handed system* or *dextral system* if a right threaded screw rotated through an angle less than 180° from **A** to **B** will advance in the direction **C** as shown in Fig.6.

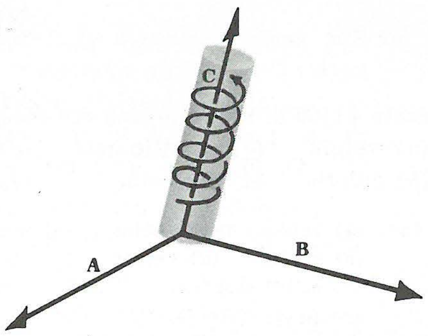


Fig. 6

COMPONENTS OF A VECTOR. Any vector **A** in 3 dimensions can be represented with initial point at the origin O of a rectangular coordinate system (Fig.7). Let (A_1, A_2, A_3) be the rectangular coordinates of the terminal point of vector **A** with initial point at O. The vectors $A_1\mathbf{i}$, $A_2\mathbf{j}$, and $A_3\mathbf{k}$ are called the *rectangular component vectors* or simply *component vectors* of **A** in the x, y and z directions respectively. A_1, A_2 and A_3 are called the *rectangular components* or simply *components* of **A** in the x, y and z directions respectively.

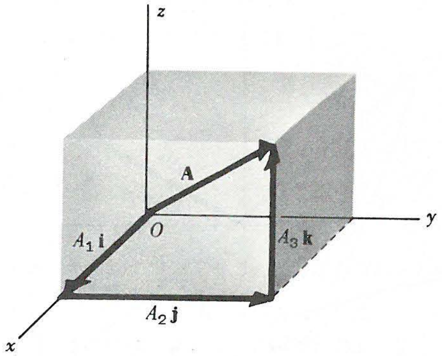


Fig. 7

The sum or resultant of $A_1\mathbf{i}$, $A_2\mathbf{j}$ and $A_3\mathbf{k}$ is the vector **A** so that we can write

$$\mathbf{A} = A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k}$$

The magnitude of **A** is $$A = |\mathbf{A}| = \sqrt{A_1^2 + A_2^2 + A_3^2}$$

In particular, the *position vector* or *radius vector* **r** from O to the point (x,y,z) is written

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

and has magnitude $r = |\mathbf{r}| = \sqrt{x^2 + y^2 + z^2}$.

SCALAR FIELD. If to each point (x,y,z) of a region R in space there corresponds a number or scalar $\phi(x,y,z)$, then ϕ is called a *scalar function of position* or *scalar point function* and we say that a *scalar field* ϕ has been defined in R.

Examples. (*1*) The temperature at any point within or on the earth's surface at a certain time defines a scalar field.

(*2*) $\phi(x,y,z) = x^3y - z^2$ defines a scalar field.

A scalar field which is independent of time is called a *stationary* or *steady-state scalar field.*

VECTOR FIELD. If to each point (x,y,z) of a region R in space there corresponds a vector $\mathbf{V}(x,y,z)$, then **V** is called a *vector function of position* or *vector point function* and we say that a *vector field* **V** has been defined in R.

Examples. (*1*) If the velocity at any point (x,y,z) within a moving fluid is known at a certain time, then a vector field is defined.

(*2*) $\mathbf{V}(x,y,z) = xy^2\mathbf{i} - 2yz^3\mathbf{j} + x^2z\mathbf{k}$ defines a vector field.

A vector field which is independent of time is called a *stationary* or *steady-state vector field.*

Chapter 2

The DOT and CROSS PRODUCT

THE DOT OR SCALAR PRODUCT of two vectors **A** and **B**, denoted by $\mathbf{A}\cdot\mathbf{B}$ (read **A** dot **B**), is defined as the product of the magnitudes of **A** and **B** and the cosine of the angle θ between them. In symbols,

$$\mathbf{A}\cdot\mathbf{B} = AB\cos\theta, \qquad 0 \leqq \theta \leqq \pi$$

Note that $\mathbf{A}\cdot\mathbf{B}$ is a scalar and not a vector.

The following laws are valid:

1. $\mathbf{A}\cdot\mathbf{B} = \mathbf{B}\cdot\mathbf{A}$ Commutative Law for Dot Products
2. $\mathbf{A}\cdot(\mathbf{B}+\mathbf{C}) = \mathbf{A}\cdot\mathbf{B} + \mathbf{A}\cdot\mathbf{C}$ Distributive Law
3. $m(\mathbf{A}\cdot\mathbf{B}) = (m\mathbf{A})\cdot\mathbf{B} = \mathbf{A}\cdot(m\mathbf{B}) = (\mathbf{A}\cdot\mathbf{B})m$, where m is a scalar.
4. $\mathbf{i}\cdot\mathbf{i} = \mathbf{j}\cdot\mathbf{j} = \mathbf{k}\cdot\mathbf{k} = 1, \quad \mathbf{i}\cdot\mathbf{j} = \mathbf{j}\cdot\mathbf{k} = \mathbf{k}\cdot\mathbf{i} = 0$
5. If $\mathbf{A} = A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k}$ and $\mathbf{B} = B_1\mathbf{i} + B_2\mathbf{j} + B_3\mathbf{k}$, then

$$\mathbf{A}\cdot\mathbf{B} = A_1B_1 + A_2B_2 + A_3B_3$$
$$\mathbf{A}\cdot\mathbf{A} = A^2 = A_1^2 + A_2^2 + A_3^2$$
$$\mathbf{B}\cdot\mathbf{B} = B^2 = B_1^2 + B_2^2 + B_3^2$$

6. If $\mathbf{A}\cdot\mathbf{B} = 0$ and **A** and **B** are not null vectors, then **A** and **B** are perpendicular.

THE CROSS OR VECTOR PRODUCT of **A** and **B** is a vector $\mathbf{C} = \mathbf{A}\times\mathbf{B}$ (read **A** cross **B**). The magnitude of $\mathbf{A}\times\mathbf{B}$ is defined as the product of the magnitudes of **A** and **B** and the sine of the angle θ between them. The direction of the vector $\mathbf{C} = \mathbf{A}\times\mathbf{B}$ is perpendicular to the plane of **A** and **B** and such that **A**, **B** and **C** form a right-handed system. In symbols,

$$\mathbf{A}\times\mathbf{B} = AB\sin\theta\,\mathbf{u}, \qquad 0 \leqq \theta \leqq \pi$$

where **u** is a unit vector indicating the direction of $\mathbf{A}\times\mathbf{B}$. If $\mathbf{A}=\mathbf{B}$, or if **A** is parallel to **B**, then $\sin\theta = 0$ and we define $\mathbf{A}\times\mathbf{B} = \mathbf{0}$.

The following laws are valid:

1. $\mathbf{A}\times\mathbf{B} = -\mathbf{B}\times\mathbf{A}$ (Commutative Law for Cross Products Fails.)
2. $\mathbf{A}\times(\mathbf{B}+\mathbf{C}) = \mathbf{A}\times\mathbf{B} + \mathbf{A}\times\mathbf{C}$ Distributive Law
3. $m(\mathbf{A}\times\mathbf{B}) = (m\mathbf{A})\times\mathbf{B} = \mathbf{A}\times(m\mathbf{B}) = (\mathbf{A}\times\mathbf{B})m$, where m is a scalar.
4. $\mathbf{i}\times\mathbf{i} = \mathbf{j}\times\mathbf{j} = \mathbf{k}\times\mathbf{k} = \mathbf{0}, \quad \mathbf{i}\times\mathbf{j} = \mathbf{k},\ \mathbf{j}\times\mathbf{k} = \mathbf{i},\ \mathbf{k}\times\mathbf{i} = \mathbf{j}$
5. If $\mathbf{A} = A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k}$ and $\mathbf{B} = B_1\mathbf{i} + B_2\mathbf{j} + B_3\mathbf{k}$, then

$$\mathbf{A}\times\mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix}$$

6. The magnitude of $\mathbf{A}\times\mathbf{B}$ is the same as the area of a parallelogram with sides $\mathbf{A}$ and $\mathbf{B}$.
7. If $\mathbf{A}\times\mathbf{B} = \mathbf{0}$, and $\mathbf{A}$ and $\mathbf{B}$ are not null vectors, then $\mathbf{A}$ and $\mathbf{B}$ are parallel.

TRIPLE PRODUCTS. Dot and cross multiplication of three vectors $\mathbf{A}, \mathbf{B}$ and $\mathbf{C}$ may produce meaningful products of the form $(\mathbf{A}\cdot\mathbf{B})\mathbf{C}$, $\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})$ and $\mathbf{A}\times(\mathbf{B}\times\mathbf{C})$. The following laws are valid:

$(A\cdot B)\times C$ = undefined

1. $(\mathbf{A}\cdot\mathbf{B})\mathbf{C} \neq \mathbf{A}(\mathbf{B}\cdot\mathbf{C})$
2. $\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \mathbf{B}\cdot(\mathbf{C}\times\mathbf{A}) = \mathbf{C}\cdot(\mathbf{A}\times\mathbf{B})$ = volume of a parallelepiped having $\mathbf{A}, \mathbf{B}$ and $\mathbf{C}$ as edges, or the negative of this volume, according as $\mathbf{A}, \mathbf{B}$ and $\mathbf{C}$ do or do not form a right-handed system. If $\mathbf{A} = A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k}$, $\mathbf{B} = B_1\mathbf{i} + B_2\mathbf{j} + B_3\mathbf{k}$ and $\mathbf{C} = C_1\mathbf{i} + C_2\mathbf{j} + C_3\mathbf{k}$, then

$$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix}$$

3. $\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) \neq (\mathbf{A}\times\mathbf{B})\times\mathbf{C}$ (Associative Law for Cross Products Fails.)
4. $\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = (\mathbf{A}\cdot\mathbf{C})\mathbf{B} - (\mathbf{A}\cdot\mathbf{B})\mathbf{C}$
   $(\mathbf{A}\times\mathbf{B})\times\mathbf{C} = (\mathbf{A}\cdot\mathbf{C})\mathbf{B} - (\mathbf{B}\cdot\mathbf{C})\mathbf{A}$

The product $\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})$ is sometimes called the *scalar triple product* or *box product* and may be denoted by $[\mathbf{ABC}]$. The product $\mathbf{A}\times(\mathbf{B}\times\mathbf{C})$ is called the *vector triple product*.

In $\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})$ parentheses are sometimes omitted and we write $\mathbf{A}\cdot\mathbf{B}\times\mathbf{C}$ (see Problem 41). However, parentheses must be used in $\mathbf{A}\times(\mathbf{B}\times\mathbf{C})$ (see Problems 29 and 47).

RECIPROCAL SETS OF VECTORS. The sets of vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ and $\mathbf{a}', \mathbf{b}', \mathbf{c}'$ are called *reciprocal sets or systems of vectors* if

$$\mathbf{a}\cdot\mathbf{a}' = \mathbf{b}\cdot\mathbf{b}' = \mathbf{c}\cdot\mathbf{c}' = 1$$

$$\mathbf{a}'\cdot\mathbf{b} = \mathbf{a}'\cdot\mathbf{c} = \mathbf{b}'\cdot\mathbf{a} = \mathbf{b}'\cdot\mathbf{c} = \mathbf{c}'\cdot\mathbf{a} = \mathbf{c}'\cdot\mathbf{b} = 0$$

The sets $\mathbf{a}, \mathbf{b}, \mathbf{c}$ and $\mathbf{a}', \mathbf{b}', \mathbf{c}'$ are reciprocal sets of vectors if and only if

$$\mathbf{a}' = \frac{\mathbf{b}\times\mathbf{c}}{\mathbf{a}\cdot\mathbf{b}\times\mathbf{c}}, \qquad \mathbf{b}' = \frac{\mathbf{c}\times\mathbf{a}}{\mathbf{a}\cdot\mathbf{b}\times\mathbf{c}}, \qquad \mathbf{c}' = \frac{\mathbf{a}\times\mathbf{b}}{\mathbf{a}\cdot\mathbf{b}\times\mathbf{c}}$$

where $\mathbf{a}\cdot\mathbf{b}\times\mathbf{c} \neq 0$. See Problems 53 and 54.

Simpson's Rule: prefixed + to minor if sum of row and column in which element lies is even. Prefix − if sum of row & column of element is odd.

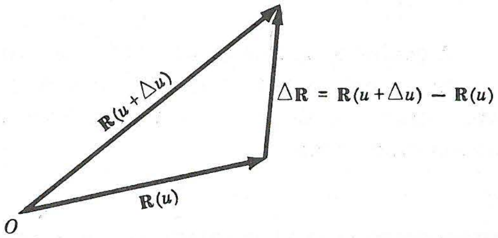
Chapter 3

VECTOR DIFFERENTIATION

ORDINARY DERIVATIVES OF VECTORS. Let $\mathbf{R}(u)$ be a vector depending on a single scalar variable u. Then

$$\frac{\Delta\mathbf{R}}{\Delta u} = \frac{\mathbf{R}(u+\Delta u) - \mathbf{R}(u)}{\Delta u}$$

where Δu denotes an increment in u (see adjoining figure).



The ordinary derivative of the vector $\mathbf{R}(u)$ with respect to the scalar u is given by

$$\frac{d\mathbf{R}}{du} = \lim_{\Delta u\to 0}\frac{\Delta\mathbf{R}}{\Delta u} = \lim_{\Delta u\to 0}\frac{\mathbf{R}(u+\Delta u) - \mathbf{R}(u)}{\Delta u}$$

if the limit exists.

Since $\dfrac{d\mathbf{R}}{du}$ is itself a vector depending on u, we can consider its derivative with respect to u. If this derivative exists it is denoted by $\dfrac{d^2\mathbf{R}}{du^2}$. In like manner higher order derivatives are described.

SPACE CURVES. If in particular $\mathbf{R}(u)$ is the position vector $\mathbf{r}(u)$ joining the origin O of a coordinate system and any point (x,y,z), then

$$\mathbf{r}(u) = x(u)\mathbf{i} + y(u)\mathbf{j} + z(u)\mathbf{k}$$

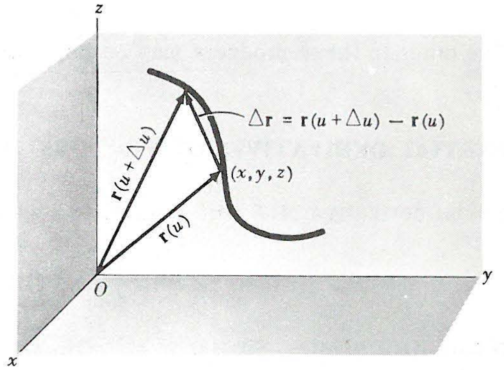
and specification of the vector function $\mathbf{r}(u)$ defines x,y and z as functions of u.

As u changes, the terminal point of $\mathbf{r}$ describes a *space curve* having parametric equations

$$x = x(u), \quad y = y(u), \quad z = z(u)$$

Then $\dfrac{\Delta\mathbf{r}}{\Delta u} = \dfrac{\mathbf{r}(u+\Delta u) - \mathbf{r}(u)}{\Delta u}$ is a vector in the direction of $\Delta\mathbf{r}$ (see adjacent figure). If $\lim\limits_{\Delta u\to 0}\dfrac{\Delta\mathbf{r}}{\Delta u} = \dfrac{d\mathbf{r}}{du}$ exists, the limit will be a vector in the direction of the tangent to the space curve at (x,y,z) and is given by

$$\frac{d\mathbf{r}}{du} = \frac{dx}{du}\mathbf{i} + \frac{dy}{du}\mathbf{j} + \frac{dz}{du}\mathbf{k}$$



If u is the time t, $\dfrac{d\mathbf{r}}{dt}$ represents the *velocity* $\mathbf{v}$ with which the terminal point of $\mathbf{r}$ describes the curve. Similarly, $\dfrac{d\mathbf{v}}{dt} = \dfrac{d^2\mathbf{r}}{dt^2}$ represents its *acceleration* $\mathbf{a}$ along the curve.

CONTINUITY AND DIFFERENTIABILITY. A scalar function $\phi(u)$ is called *continuous* at u if $\lim_{\Delta u \to 0} \phi(u+\Delta u) = \phi(u)$. Equivalently, $\phi(u)$ is continuous at u if for each positive number ϵ we can find some positive number δ such that

$$|\phi(u+\Delta u) - \phi(u)| < \epsilon \quad \text{whenever} \quad |\Delta u| < \delta .$$

A vector function $\mathbf{R}(u) = R_1(u)\mathbf{i} + R_2(u)\mathbf{j} + R_3(u)\mathbf{k}$ is called *continuous* at u if the three scalar functions $R_1(u)$, $R_2(u)$ and $R_3(u)$ are continuous at u or if $\lim_{\Delta u \to 0} \mathbf{R}(u+\Delta u) = \mathbf{R}(u)$. Equivalently, $\mathbf{R}(u)$ is continuous at u if for each positive number ϵ we can find some positive number δ such that

$$|\mathbf{R}(u+\Delta u) - \mathbf{R}(u)| < \epsilon \quad \text{whenever} \quad |\Delta u| < \delta .$$

A scalar or vector function of u is called *differentiable of order n* if its nth derivative exists. A function which is differentiable is necessarily continuous but the converse is not true. Unless otherwise stated we assume that all functions considered are differentiable to any order needed in a particular discussion.

DIFFERENTIATION FORMULAS. If $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ are differentiable vector functions of a scalar u, and ϕ is a differentiable scalar function of u, then

1. $$\frac{d}{du}(\mathbf{A}+\mathbf{B}) = \frac{d\mathbf{A}}{du} + \frac{d\mathbf{B}}{du}$$

2. $$\frac{d}{du}(\mathbf{A}\cdot\mathbf{B}) = \mathbf{A}\cdot\frac{d\mathbf{B}}{du} + \frac{d\mathbf{A}}{du}\cdot\mathbf{B}$$

3. $$\frac{d}{du}(\mathbf{A}\times\mathbf{B}) = \mathbf{A}\times\frac{d\mathbf{B}}{du} + \frac{d\mathbf{A}}{du}\times\mathbf{B}$$

4. $$\frac{d}{du}(\phi\mathbf{A}) = \phi\frac{d\mathbf{A}}{du} + \frac{d\phi}{du}\mathbf{A}$$

5. $$\frac{d}{du}(\mathbf{A}\cdot\mathbf{B}\times\mathbf{C}) = \mathbf{A}\cdot\mathbf{B}\times\frac{d\mathbf{C}}{du} + \mathbf{A}\cdot\frac{d\mathbf{B}}{du}\times\mathbf{C} + \frac{d\mathbf{A}}{du}\cdot\mathbf{B}\times\mathbf{C}$$

6. $$\frac{d}{du}\{\mathbf{A}\times(\mathbf{B}\times\mathbf{C})\} = \mathbf{A}\times(\mathbf{B}\times\frac{d\mathbf{C}}{du}) + \mathbf{A}\times(\frac{d\mathbf{B}}{du}\times\mathbf{C}) + \frac{d\mathbf{A}}{du}\times(\mathbf{B}\times\mathbf{C})$$

The order in these products may be important.

PARTIAL DERIVATIVES OF VECTORS. If $\mathbf{A}$ is a vector depending on more than one scalar variable, say x, y, z for example, then we write $\mathbf{A} = \mathbf{A}(x,y,z)$. The partial derivative of $\mathbf{A}$ with respect to x is defined as

$$\frac{\partial\mathbf{A}}{\partial x} = \lim_{\Delta x \to 0} \frac{\mathbf{A}(x+\Delta x, y, z) - \mathbf{A}(x,y,z)}{\Delta x}$$

if this limit exists. Similarly,

$$\frac{\partial\mathbf{A}}{\partial y} = \lim_{\Delta y \to 0} \frac{\mathbf{A}(x, y+\Delta y, z) - \mathbf{A}(x,y,z)}{\Delta y}$$

$$\frac{\partial\mathbf{A}}{\partial z} = \lim_{\Delta z \to 0} \frac{\mathbf{A}(x,y, z+\Delta z) - \mathbf{A}(x,y,z)}{\Delta z}$$

are the partial derivatives of **A** with respect to y and z respectively if these limits exist.

The remarks on continuity and differentiability for functions of one variable can be extended to functions of two or more variables. For example, $\phi(x,y)$ is called continuous at (x,y) if $\lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}} \phi(x+\Delta x,\ y+\Delta y) = \phi(x,y)$, or if for each positive number ϵ we can find some positive number δ such that $|\phi(x+\Delta x,\ y+\Delta y) - \phi(x,y)| < \epsilon$ whenever $|\Delta x| < \delta$ and $|\Delta y| < \delta$. Similar definitions hold for vector functions.

For functions of two or more variables we use the term *differentiable* to mean that the function has continuous first partial derivatives. (The term is used by others in a slightly weaker sense.)

Higher derivatives can be defined as in the calculus. Thus, for example,

$$\frac{\partial^2 \mathbf{A}}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial \mathbf{A}}{\partial x}\right), \quad \frac{\partial^2 \mathbf{A}}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial \mathbf{A}}{\partial y}\right), \quad \frac{\partial^2 \mathbf{A}}{\partial z^2} = \frac{\partial}{\partial z}\left(\frac{\partial \mathbf{A}}{\partial z}\right)$$

$$\frac{\partial^2 \mathbf{A}}{\partial x\,\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial \mathbf{A}}{\partial y}\right), \quad \frac{\partial^2 \mathbf{A}}{\partial y\,\partial x} = \frac{\partial}{\partial y}\left(\frac{\partial \mathbf{A}}{\partial x}\right), \quad \frac{\partial^3 \mathbf{A}}{\partial x\,\partial z^2} = \frac{\partial}{\partial x}\left(\frac{\partial^2 \mathbf{A}}{\partial z^2}\right)$$

If **A** has continuous partial derivatives of the second order at least, then $\dfrac{\partial^2 \mathbf{A}}{\partial x\,\partial y} = \dfrac{\partial^2 \mathbf{A}}{\partial y\,\partial x}$, i.e. the order of differentiation does not matter.

Rules for partial differentiation of vectors are similar to those used in elementary calculus for scalar functions. Thus if **A** and **B** are functions of x,y,z then, for example,

1. $$\frac{\partial}{\partial x}(\mathbf{A}\cdot\mathbf{B}) = \mathbf{A}\cdot\frac{\partial \mathbf{B}}{\partial x} + \frac{\partial \mathbf{A}}{\partial x}\cdot\mathbf{B}$$

2. $$\frac{\partial}{\partial x}(\mathbf{A}\times\mathbf{B}) = \mathbf{A}\times\frac{\partial \mathbf{B}}{\partial x} + \frac{\partial \mathbf{A}}{\partial x}\times\mathbf{B}$$

3. $$\begin{aligned}\frac{\partial^2}{\partial y\,\partial x}(\mathbf{A}\cdot\mathbf{B}) &= \frac{\partial}{\partial y}\left\{\frac{\partial}{\partial x}(\mathbf{A}\cdot\mathbf{B})\right\} = \frac{\partial}{\partial y}\left\{\mathbf{A}\cdot\frac{\partial \mathbf{B}}{\partial x} + \frac{\partial \mathbf{A}}{\partial x}\cdot\mathbf{B}\right\}\\ &= \mathbf{A}\cdot\frac{\partial^2 \mathbf{B}}{\partial y\,\partial x} + \frac{\partial \mathbf{A}}{\partial y}\cdot\frac{\partial \mathbf{B}}{\partial x} + \frac{\partial \mathbf{A}}{\partial x}\cdot\frac{\partial \mathbf{B}}{\partial y} + \frac{\partial^2 \mathbf{A}}{\partial y\,\partial x}\cdot\mathbf{B}, \quad \text{etc.}\end{aligned}$$

DIFFERENTIALS OF VECTORS follow rules similar to those of elementary calculus. For example,

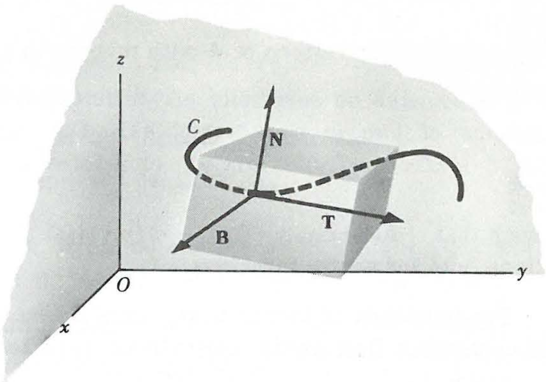
1. If $\mathbf{A} = A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k}$, then $d\mathbf{A} = dA_1\mathbf{i} + dA_2\mathbf{j} + dA_3\mathbf{k}$

2. $d(\mathbf{A}\cdot\mathbf{B}) = \mathbf{A}\cdot d\mathbf{B} + d\mathbf{A}\cdot\mathbf{B}$

3. $d(\mathbf{A}\times\mathbf{B}) = \mathbf{A}\times d\mathbf{B} + d\mathbf{A}\times\mathbf{B}$

4. If $\mathbf{A} = \mathbf{A}(x,y,z)$, then $d\mathbf{A} = \dfrac{\partial \mathbf{A}}{\partial x}dx + \dfrac{\partial \mathbf{A}}{\partial y}dy + \dfrac{\partial \mathbf{A}}{\partial z}dz$, etc.

DIFFERENTIAL GEOMETRY involves a study of space curves and surfaces. If C is a space curve defined by the function $\mathbf{r}(u)$, then we have seen that $\dfrac{d\mathbf{r}}{du}$ is a vector in the direction of the tangent to C. If the scalar u is taken as the arc length s measured from some fixed point on C, then $\dfrac{d\mathbf{r}}{ds}$ is a unit tangent vector to C and is denoted by **T** (see diagram below). The



rate at which **T** changes with respect to s is a measure of the curvature of C and is given by $\frac{d\mathbf{T}}{ds}$. The direction of $\frac{d\mathbf{T}}{ds}$ at any given point on C is normal to the curve at that point (see Problem 9). If **N** is a unit vector in this normal direction, it is called the *principal normal* to the curve. Then $\frac{d\mathbf{T}}{ds} = \kappa\mathbf{N}$, where κ is called the *curvature* of C at the specified point. The quantity $\rho = 1/\kappa$ is called the *radius of curvature.*

A unit vector **B** perpendicular to the plane of **T** and **N** and such that $\mathbf{B} = \mathbf{T}\times\mathbf{N}$, is called the *binormal* to the curve. It follows that directions **T**, **N**, **B** form a localized right-handed rectangular coordinate system at any specified point of C. This coordinate system is called the *trihedral* or *triad* at the point. As s changes, the coordinate system moves and is known as the *moving trihedral.*

A set of relations involving derivatives of the fundamental vectors **T**, **N** and **B** is known collectively as the *Frenet-Serret formulas* given by

$$\frac{d\mathbf{T}}{ds} = \kappa\mathbf{N}, \qquad \frac{d\mathbf{N}}{ds} = \tau\mathbf{B} - \kappa\mathbf{T}, \qquad \frac{d\mathbf{B}}{ds} = -\tau\mathbf{N}$$

where τ is a scalar called the *torsion.* The quantity $\sigma = 1/\tau$ is called the *radius of torsion.*

The *osculating plane* to a curve at a point P is the plane containing the tangent and principal normal at P. The *normal plane* is the plane through P perpendicular to the tangent. The *rectifying plane* is the plane through P which is perpendicular to the principal normal.

MECHANICS often includes a study of the motion of particles along curves, this study being known as *kinematics*. In this connection some of the results of differential geometry can be of value.

A study of forces on moving objects is considered in *dynamics*. Fundamental to this study is Newton's famous law which states that if **F** is the net force acting on an object of mass m moving with velocity **v**, then

$$\mathbf{F} = \frac{d}{dt}(m\mathbf{v})$$

where $m\mathbf{v}$ is the momentum of the object. If m is constant this becomes $\mathbf{F} = m\frac{d\mathbf{v}}{dt} = m\mathbf{a}$, where **a** is the acceleration of the object.

Chapter 4

GRADIENT, DIVERGENCE and CURL

THE VECTOR DIFFERENTIAL OPERATOR DEL, written ∇, is defined by

$$\nabla \equiv \frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k} \equiv \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}$$

This vector operator possesses properties analogous to those of ordinary vectors. It is useful in defining three quantities which arise in practical applications and are known as the *gradient*, the *divergence* and the *curl*. The operator ∇ is also known as *nabla*.

THE GRADIENT. Let $\phi(x,y,z)$ be defined and differentiable at each point (x,y,z) in a certain region of space (i.e. ϕ defines a differentiable scalar field). Then the *gradient* of ϕ, written $\nabla\phi$ or grad ϕ, is defined by

$$\nabla\phi = \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right)\phi = \frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} + \frac{\partial\phi}{\partial z}\mathbf{k}$$

Note that $\nabla\phi$ defines a vector field.

The component of $\nabla\phi$ in the direction of a unit vector $\mathbf{a}$ is given by $\nabla\phi \cdot \mathbf{a}$ and is called the *directional derivative* of ϕ in the direction $\mathbf{a}$. Physically, this is the rate of change of ϕ at (x,y,z) in the direction $\mathbf{a}$.

$du = \nabla u \cdot dr$

THE DIVERGENCE. Let $\mathbf{V}(x,y,z) = V_1\mathbf{i} + V_2\mathbf{j} + V_3\mathbf{k}$ be defined and differentiable at each point (x,y,z) in a certain region of space (i.e. $\mathbf{V}$ defines a differentiable vector field). Then the *divergence* of $\mathbf{V}$, written $\nabla\cdot\mathbf{V}$ or div $\mathbf{V}$, is defined by

$$\begin{aligned}\nabla\cdot\mathbf{V} &= \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right)\cdot(V_1\mathbf{i} + V_2\mathbf{j} + V_3\mathbf{k}) \\ &= \frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y} + \frac{\partial V_3}{\partial z}\end{aligned}$$

Note the analogy with $\mathbf{A}\cdot\mathbf{B} = A_1B_1 + A_2B_2 + A_3B_3$. Also note that $\nabla\cdot\mathbf{V} \neq \mathbf{V}\cdot\nabla$.

THE CURL. If $\mathbf{V}(x,y,z)$ is a differentiable vector field then the *curl* or *rotation* of $\mathbf{V}$, written $\nabla\times\mathbf{V}$, curl $\mathbf{V}$ or rot $\mathbf{V}$, is defined by

$$\begin{aligned}\nabla\times\mathbf{V} &= \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right)\times(V_1\mathbf{i} + V_2\mathbf{j} + V_3\mathbf{k}) \\ &= \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ V_1 & V_2 & V_3\end{vmatrix}\end{aligned}$$

$$= \begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_2 & V_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ V_1 & V_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ V_1 & V_2 \end{vmatrix} \mathbf{k}$$

$$= \left(\frac{\partial V_3}{\partial y} - \frac{\partial V_2}{\partial z}\right)\mathbf{i} + \left(\frac{\partial V_1}{\partial z} - \frac{\partial V_3}{\partial x}\right)\mathbf{j} + \left(\frac{\partial V_2}{\partial x} - \frac{\partial V_1}{\partial y}\right)\mathbf{k}$$

Note that in the expansion of the determinant the operators $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$ must *precede* V_1, V_2, V_3 .

FORMULAS INVOLVING ∇. If $\mathbf{A}$ and $\mathbf{B}$ are differentiable vector functions, and ϕ and ψ are differentiable scalar functions of position (x, y, z), then

1. $\nabla(\phi + \psi) = \nabla\phi + \nabla\psi$ or $\operatorname{grad}(\phi + \psi) = \operatorname{grad}\phi + \operatorname{grad}\psi$

2. $\nabla\cdot(\mathbf{A}+\mathbf{B}) = \nabla\cdot\mathbf{A} + \nabla\cdot\mathbf{B}$ or $\operatorname{div}(\mathbf{A}+\mathbf{B}) = \operatorname{div}\mathbf{A} + \operatorname{div}\mathbf{B}$

3. $\nabla\times(\mathbf{A}+\mathbf{B}) = \nabla\times\mathbf{A} + \nabla\times\mathbf{B}$ or $\operatorname{curl}(\mathbf{A}+\mathbf{B}) = \operatorname{curl}\mathbf{A} + \operatorname{curl}\mathbf{B}$

4. $\nabla\cdot(\phi\mathbf{A}) = (\nabla\phi)\cdot\mathbf{A} + \phi(\nabla\cdot\mathbf{A})$

5. $\nabla\times(\phi\mathbf{A}) = (\nabla\phi)\times\mathbf{A} + \phi(\nabla\times\mathbf{A})$

6. $\nabla\cdot(\mathbf{A}\times\mathbf{B}) = \mathbf{B}\cdot(\nabla\times\mathbf{A}) - \mathbf{A}\cdot(\nabla\times\mathbf{B})$

7. $\nabla\times(\mathbf{A}\times\mathbf{B}) = (\mathbf{B}\cdot\nabla)\mathbf{A} - \mathbf{B}(\nabla\cdot\mathbf{A}) - (\mathbf{A}\cdot\nabla)\mathbf{B} + \mathbf{A}(\nabla\cdot\mathbf{B})$

8. $\nabla(\mathbf{A}\cdot\mathbf{B}) = (\mathbf{B}\cdot\nabla)\mathbf{A} + (\mathbf{A}\cdot\nabla)\mathbf{B} + \mathbf{B}\times(\nabla\times\mathbf{A}) + \mathbf{A}\times(\nabla\times\mathbf{B})$

9. $\nabla\cdot(\nabla\phi) \equiv \nabla^2\phi \equiv \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}$

where $\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ is called the *Laplacian operator.*

10. $\nabla\times(\nabla\phi) = \mathbf{0}$. The curl of the gradient of ϕ is zero.

11. $\nabla\cdot(\nabla\times\mathbf{A}) = 0$. The divergence of the curl of $\mathbf{A}$ is zero.

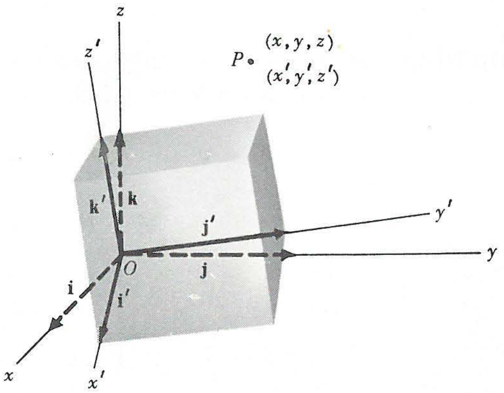
12. $\nabla\times(\nabla\times\mathbf{A}) = \nabla(\nabla\cdot\mathbf{A}) - \nabla^2\mathbf{A}$

In Formulas 9-12, it is supposed that ϕ and $\mathbf{A}$ have continuous second partial derivatives.

INVARIANCE. Consider two rectangular coordinate systems or frames of reference xyz and $x'y'z'$ (see figure below) having the same origin O but with axes rotated with respect to each other.

A point P in space has coordinates (x, y, z) or (x', y', z') relative to these coordinate systems. The equations of transformation between coordinates or the *coordinate transformations* are given by

$$(1) \qquad \begin{aligned} x' &= l_{11}x + l_{12}y + l_{13}z \\ y' &= l_{21}x + l_{22}y + l_{23}z \\ z' &= l_{31}x + l_{32}y + l_{33}z \end{aligned}$$



where l_{jk}, $j, k = 1,2,3$, represent direction cosines of the x', y' and z' axes with respect to the x, y, and

z axes (see Problem 38). In case the origins of the two coordinate systems are not coincident the equations of transformation become

$$(2)\qquad \begin{cases} x' = l_{11}x + l_{12}y + l_{13}z + a_1' \\ y' = l_{21}x + l_{22}y + l_{23}z + a_2' \\ z' = l_{31}x + l_{32}y + l_{33}z + a_3' \end{cases}$$

where origin O of the xyz coordinate system is located at (a_1', a_2', a_3') relative to the $x'y'z'$ coordinate system.

The transformation equations *(1)* define a *pure rotation*, while equations *(2)* define a *rotation plus translation*. Any rigid body motion has the effect of a translation followed by a rotation. The transformation *(1)* is also called an *orthogonal transformation*. A general linear transformation is called an *affine transformation*.

Physically a scalar point function or scalar field $\phi(x,y,z)$ evaluated at a particular point should be independent of the coordinates of the point. Thus the temperature at a point is not dependent on whether coordinates (x,y,z) or (x',y',z') are used. Then if $\phi(x,y,z)$ is the temperature at point P with coordinates (x,y,z) while $\phi'(x',y',z')$ is the temperature at the same point P with coordinates (x',y',z'), we must have $\phi(x,y,z) = \phi'(x',y',z')$. If $\phi(x,y,z) = \phi'(x',y',z')$, where x,y,z and x',y',z' are related by the transformation equations *(1)* or *(2)*, we call $\phi(x,y,z)$ an *invariant* with respect to the transformation. For example, $x^2+y^2+z^2$ is invariant under the transformation of rotation *(1)*, since $x^2+y^2+z^2 = x'^2 + y'^2 + z'^2$.

Similarly, a vector point function or vector field $\mathbf{A}(x,y,z)$ is called an *invariant* if $\mathbf{A}(x,y,z) = \mathbf{A}'(x',y',z')$. This will be true if

$$A_1(x,y,z)\mathbf{i} + A_2(x,y,z)\mathbf{j} + A_3(x,y,z)\mathbf{k} = A_1'(x',y',z')\mathbf{i}' + A_2'(x',y',z')\mathbf{j}' + A_3'(x',y',z')\mathbf{k}'$$

In Chap. 7 and 8, more general transformations are considered and the above concepts are extended.

It can be shown (see Problem 41) that the gradient of an invariant scalar field is an invariant vector field with respect to the transformations *(1)* or *(2)*. Similarly, the divergence and curl of an invariant vector field are invariant under this transformation.

SOLVED PROBLEMS

THE GRADIENT

1. If $\phi(x,y,z) = 3x^2y - y^3z^2$, find $\nabla\phi$ (or grad ϕ) at the point $(1,-2,-1)$.

$$\begin{aligned}
\nabla\phi &= \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right)(3x^2y - y^3z^2) \\
&= \mathbf{i}\frac{\partial}{\partial x}(3x^2y - y^3z^2) + \mathbf{j}\frac{\partial}{\partial y}(3x^2y - y^3z^2) + \mathbf{k}\frac{\partial}{\partial z}(3x^2y - y^3z^2) \\
&= 6xy\,\mathbf{i} + (3x^2 - 3y^2z^2)\mathbf{j} - 2y^3z\,\mathbf{k} \\
&= 6(1)(-2)\mathbf{i} + \{3(1)^2 - 3(-2)^2(-1)^2\}\mathbf{j} - 2(-2)^3(-1)\mathbf{k} \\
&= -12\mathbf{i} - 9\mathbf{j} - 16\mathbf{k}
\end{aligned}$$

CHAPTER 1

Origin of Differential Equations

A DIFFERENTIAL EQUATION is an equation which involves derivatives. For example,

1) $\frac{dy}{dx} = x + 5$

2) $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = 0$

3) $xy' + y = 3$

4) $y''' + 2(y'')^2 + y' = \cos x$

5) $(y'')^2 + (y')^3 + 3y = x^2$

6) $\frac{\partial z}{\partial x} = z + x\frac{\partial z}{\partial y}$

7) $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = x^2 + y.$

If there is a single independent variable, as in 1)-5), the derivatives are ordinary derivatives and the equation is called an *ordinary differential equation*.

If there are two or more independent variables, as in 6)-7), the derivatives are partial derivatives and the equation is called a *partial differential equation*.

The *order* of a differential equation is the order of the highest derivative which occurs. Equations 1), 3), and 6) are of the first order; 2), 5), and 7) are of the second order; and 4) is of the third order.

The *degree* of a differential equation which can be written as a polynomial in the derivatives is the degree of the highest ordered derivative which then occurs. All of the above examples are of the first degree except 5) which is of the second degree.

A discussion of partial differential equations will be given in Chapter 28. For the present, only ordinary differential equations with a single dependent variable will be considered.

ORIGIN OF DIFFERENTIAL EQUATIONS.

a) Geometric Problems. See Problems 1 and 2 below.

b) Physical Problems. See Problems 3 and 4 below.

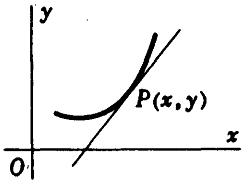
c) Primitives. A relation between the variables which involves n essential arbitrary constants, as $y = x^4 + Cx$ or $y = Ax^2 + Bx$, is called a *primitive*. The n constants, always indicated by capital letters here, are called *essential* if they cannot be replaced by a smaller number of constants. See Problem 5.

In general, a primitive involving n essential arbitrary constants will give rise to a differential equation, of order n, free of arbitrary constants. This equation is obtained by eliminating the n constants between the $(n+1)$ equations consisting of the primitive and the n equations obtained by differentiating the primitive n times with respect to the independent variable. See Problems 6-14 below.

SOLVED PROBLEMS

1. A curve is defined by the condition that at each of its points (x,y), its slope dy/dx is equal to twice the sum of the coordinates of the point. Express the condition by means of a differential equation.

   The differential equation representing the condition is $\frac{dy}{dx} = 2(x+y)$.



2. A curve is defined by the condition that the sum of the x- and y-intercepts of its tangents is always equal to 2. Express the condition by means of a differential equation.

   The equation of the tangent at (x,y) on the curve is $Y-y = \frac{dy}{dx}(X-x)$ and the x- and y-intercepts are respectively $X = x-y\frac{dx}{dy}$ and $Y = y-x\frac{dy}{dx}$. The differential equation representing the condition is $X+Y = x-y\frac{dx}{dy} + y-x\frac{dy}{dx} = 2$ or $x(\frac{dy}{dx})^2 - (x+y-2)\frac{dy}{dx} + y = 0$.

3. One hundred grams of cane sugar in water are being converted into dextrose at a rate which is proportional to the amount unconverted. Find the differential equation expressing the rate of conversion after t minutes.

   Let q denote the number of grams converted in t minutes. Then $(100-q)$ is the number of grams unconverted and the rate of conversion is given by $\frac{dq}{dt} = k(100-q)$, k being the constant of proportionality.

4. A particle of mass m moves along a straight line (the x-axis) while subject to 1) a force proportional to its displacement x from a fixed point O in its path and directed toward O and 2) a resisting force proportional to its velocity. Express the total force as a differential equation.

   The first force may be represented by $-k_1x$ and the second by $-k_2\frac{dx}{dt}$, where k_1 and k_2 are factors of proportionality.

   The total force (mass × acceleration) is given by $m\frac{d^2x}{dt^2} = -k_1x - k_2\frac{dx}{dt}$.

5. In each of the equations *a*) $y = x^2+A+B$, *b*) $y = Ae^{x+B}$, *c*) $y = A + \ln Bx$ show that only one of the two arbitrary constants is essential.

   a) Since $A+B$ is no more than a single arbitrary constant, only one essential arbitrary constant is involved.

   b) $y = Ae^{x+B} = Ae^xe^B$, and Ae^B is no more than a single arbitrary constant.

   c) $y = A + \ln Bx = A + \ln B + \ln x$, and $(A + \ln B)$ is no more than a single constant.

6. Obtain the differential equation associated with the primitive $y = Ax^2+Bx+C$.

   Since there are three arbitrary constants, we consider the four equations

$$y = Ax^2+Bx+C, \qquad \frac{dy}{dx} = 2Ax+B, \qquad \frac{d^2y}{dx^2} = 2A, \qquad \frac{d^3y}{dx^3} = 0.$$

   The last of these $\frac{d^3y}{dx^3}$, being free of arbitrary constants and of the proper order, is the required equation.

Note that the constants could not have been eliminated between the first three of the above equations. Note also that the primitive can be obtained readily from the differential equation by integration.

7. Obtain the differential equation associated with the primitive $x^2y^3 + x^3y^5 = C$.

Differentiating once with respect to x, we obtain $(2xy^3 + 3x^2y^2\frac{dy}{dx}) + (3x^2y^5 + 5x^3y^4\frac{dy}{dx}) = 0$

or, when $xy \neq 0$, $(2y + 3x\frac{dy}{dx}) + xy^2(3y + 5x\frac{dy}{dx}) = 0$ as the required equation.

When written in differential notation, these equations are

$$1) \quad (2xy^3\,dx + 3x^2y^2\,dy) + (3x^2y^5\,dx + 5x^3y^4\,dy) = 0$$

and

$$2) \quad (2y\,dx + 3x\,dy) + xy^2(3y\,dx + 5x\,dy) = 0.$$

Note that the primitive can be obtained readily from 1) by integration but not so readily from 2). Thus, to obtain the primitive when 2) is given, it is necessary to determine the factor xy^2 which was removed from 1).

8. Obtain the differential equation associated with the primitive $y = A\cos ax + B\sin ax$, A and B being arbitrary constants, and a being a fixed constant.

Here
$$\frac{dy}{dx} = -Aa\sin ax + Ba\cos ax$$

and
$$\frac{d^2y}{dx^2} = -Aa^2\cos ax - Ba^2\sin ax = -a^2(A\cos ax + B\sin ax) = -a^2y.$$

The required differential equation is $\frac{d^2y}{dx^2} + a^2y = 0$.

9. Obtain the differential equation associated with the primitive $y = Ae^{2x} + Be^x + C$.

Here $\frac{dy}{dx} = 2Ae^{2x} + Be^x$, $\frac{d^2y}{dx^2} = 4Ae^{2x} + Be^x$, $\frac{d^3y}{dx^3} = 8Ae^{2x} + Be^x$.

Then $\frac{d^3y}{dx^3} - \frac{d^2y}{dx^2} = 4Ae^{2x}$, $\frac{d^2y}{dx^2} - \frac{dy}{dx} = 2Ae^{2x}$, and $\frac{d^3y}{dx^3} - \frac{d^2y}{dx^2} = 2(\frac{d^2y}{dx^2} - \frac{dy}{dx})$.

The required equation is $\frac{d^3y}{dx^3} - 3\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = 0$.

10. Obtain the differential equation associated with the primitive $y = C_1e^{3x} + C_2e^{2x} + C_3e^x$.

Here $\frac{dy}{dx} = 3C_1e^{3x} + 2C_2e^{2x} + C_3e^x$, $\frac{d^2y}{dx^2} = 9C_1e^{3x} + 4C_2e^{2x} + C_3e^x$,

and $\frac{d^3y}{dx^3} = 27C_1e^{3x} + 8C_2e^{2x} + C_3e^x$.

The elimination of the constants by elementary methods is somewhat tedious. If three of the equations are solved for C_1, C_2, C_3, using determinants, and these substituted in the fourth equation, the result may be put in the form (called the eliminant):

$$\begin{vmatrix} e^{3x} & e^{2x} & e^{x} & y \\ 3e^{3x} & 2e^{2x} & e^{x} & y' \\ 9e^{3x} & 4e^{2x} & e^{x} & y'' \\ 27e^{3x} & 8e^{2x} & e^{x} & y''' \end{vmatrix} = e^{6x}\begin{vmatrix} 1 & 1 & 1 & y \\ 3 & 2 & 1 & y' \\ 9 & 4 & 1 & y'' \\ 27 & 8 & 1 & y''' \end{vmatrix} = e^{6x}(-2y''' + 12y'' - 22y' + 12y) = 0.$$

The required differential equation is $\dfrac{d^3y}{dx^3} - 6\dfrac{d^2y}{dx^2} + 11\dfrac{dy}{dx} - 6y = 0.$

11. Obtain the differential equation associated with the primitive $y = Cx^2 + C^2$.

Since $\dfrac{dy}{dx} = 2Cx$, $C = \dfrac{1}{2x}\dfrac{dy}{dx}$ and $y = Cx^2 + C^2 = \dfrac{1}{2x}\dfrac{dy}{dx}x^2 + \dfrac{1}{4x^2}(\dfrac{dy}{dx})^2$.

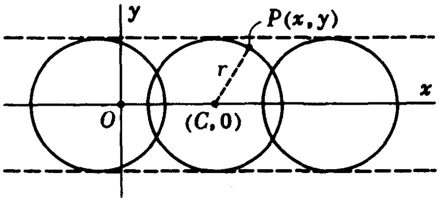
The required differential equation is $(\dfrac{dy}{dx})^2 + 2x^3\dfrac{dy}{dx} - 4x^2y = 0$.

Note. The primitive involves one arbitrary constant of degree two and the resulting differential equation is of order one and degree two.

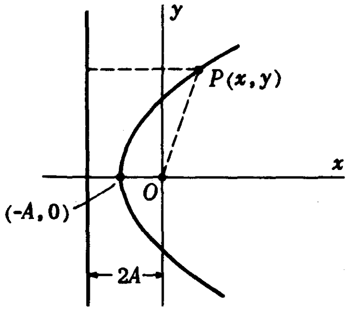
12. Find the differential equation of the family of circles of fixed radius r with centers on the x-axis.

The equation of the family is $(x-C)^2 + y^2 = r^2$, C being an arbitrary constant.

Then $(x-C) + y\dfrac{dy}{dx} = 0$, $x - C = -y\dfrac{dy}{dx}$, and the differential equation is $y^2(\dfrac{dy}{dx})^2 + y^2 = r^2$.

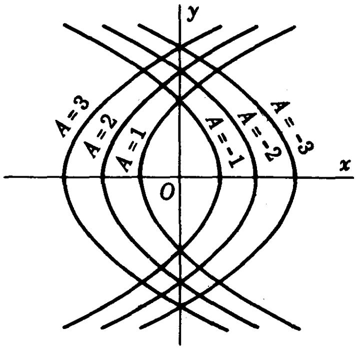


13. Find the differential equation of the family of parabolas with foci at the origin and axes along the x-axis.



$x^2 + y^2 = (2A + x)^2$

$y^2 = 4A(A + x)$



$y^2 = 4A(A + x)$

The equation of the family of parabolas is $y^2 = 4A(A + x)$.
Then $yy' = 2A$, $A = \frac{1}{2}yy'$, and $y^2 = 2yy'(\frac{1}{2}yy' + x)$.

The required equation is $y(\dfrac{dy}{dx})^2 + 2x\dfrac{dy}{dx} - y = 0$.

14. Form the differential equation representing all tangents to the parabola $y^2 = 2x$.

At any point (A,B) on the parabola, the equation of the tangent is $y-B = (x-A)/B$ or, since $A = \frac{1}{2}B^2$, $By = x + \frac{1}{2}B^2$. Eliminating B between this and $By' = 1$, obtained by differentiation with respect to x, we have as the required differential equation $2x(y')^2 - 2yy' + 1 = 0$.

SUPPLEMENTARY PROBLEMS

15. Classify each of the following equations as to order and degree.

a) $dy + (xy - \cos x)dx = 0$ *Ans.* Order one; degree one

b) $L\dfrac{d^2Q}{dt^2} + R\dfrac{dQ}{dt} + \dfrac{Q}{C} = 0$ *Ans.* Order two; degree one

c) $y''' + xy'' + 2y(y')^2 + xy = 0$ *Ans.* Order three; degree one

d) $\dfrac{d^2v}{dx^2}\dfrac{dv}{dx} + x(\dfrac{dv}{dx})^2 + v = 0$ *Ans.* Order two; degree one

e) $(\dfrac{d^3w}{dv^3})^2 - (\dfrac{d^2w}{dv^2})^4 + vw = 0$ *Ans.* Order three; degree two

f) $e^{y'''} - xy'' + y = 0$ *Ans.* Order three; degree does not apply

g) $\sqrt{\rho' + \rho} = \sin\theta$ *Ans.* Order one; degree one

h) $y' + x = (y - xy')^{-3}$ *Ans.* Order one; degree four

i) $\dfrac{d^2\rho}{d\theta^2} = \sqrt[4]{\rho + (\dfrac{d\rho}{d\theta})^2}$ *Ans.* Order two; degree four

16. Write the differential equation for each of the curves determined by the given conditions.

a) At each point (x,y) the slope of the tangent is equal to the square of the abscissa of the point. *Ans.* $y' = x^2$

b) At each point (x,y) the length of the subtangent is equal to the sum of the coordinates of the point. *Ans.* $y/y' = x + y$ or $(x+y)y' = y$

c) The segment joining $P(x,y)$ and the point of intersection of the normal at P with the x-axis is bisected by the y-axis. *Ans.* $y + x\dfrac{dx}{dy} = \frac{1}{2}y$ or $yy' + 2x = 0$

d) At each point (ρ,θ) the tangent of the angle between the radius vector and the tangent is equal to 1/3 the tangent of the vectorial angle. *Ans.* $\rho\dfrac{d\theta}{d\rho} = \dfrac{1}{3}\tan\theta$

e) The area bounded by the arc of a curve, the x-axis, and two ordinates, one fixed and one variable, is equal to twice the length of the arc between the ordinates.

Hint: $\displaystyle\int_a^x y\,dx = 2\int_a^x \sqrt{1+(y')^2}\,dx$. *Ans.* $y = 2\sqrt{1+(y')^2}$

17. Express each of the following physical statements in differential equation form.

a) Radium decomposes at a rate proportional to the amount Q present. *Ans.* $dQ/dt = -kQ$

b) The population P of a city increases at a rate proportional to the population and to the difference between 200,000 and the population. *Ans.* $dP/dt = kP(200{,}000 - P)$

c) For a certain substance the rate of change of vapor pressure (P) with respect to temperature (T) is proportional to the vapor pressure and inversely proportional to the square of the temperature. *Ans.* $dP/dT = kP/T^2$

d) The potential difference E across an element of inductance L is equal to the product of L and the time rate of change of the current i in the inductance. *Ans.* $E = L\dfrac{di}{dt}$

e) Mass × acceleration = net force. *Ans.* $m\dfrac{dv}{dt} = F$ or $m\dfrac{d^2s}{dt^2} = F$

18. Obtain the differential equation associated with the given primitive, A and B being arbitrary constants.

a) $y = Ax$ *Ans.* $y' = y/x$

b) $y = Ax + B$ *Ans.* $y'' = 0$

c) $y = e^{x+A} = Be^x$ *Ans.* $y' = y$

d) $y = A \sin x$ *Ans.* $y' = y \cot x$

e) $y = \sin(x+A)$ *Ans.* $(y')^2 = 1 - y^2$

f) $y = Ae^x + B$ *Ans.* $y'' = y'$

g) $x = A \sin(y+B)$ *Ans.* $y'' = x(y')^3$

h) $\ln y = Ax^2 + B$ *Ans.* $xyy'' - yy' - x(y')^2 = 0$

19. Find the differential equation of the family of circles of variable radii r with centers on the x-axis. (Compare with Problem 12.)
Hint: $(x-A)^2 + y^2 = r^2$, A and r being arbitrary constants. *Ans.* $yy'' + (y')^2 + 1 = 0$

20. Find the differential equation of the family of cardiods $\rho = a(1 - \cos\theta)$.
Ans. $(1 - \cos\theta)d\rho = \rho \sin\theta \, d\theta$

21. Find the differential equation of all straight lines at a unit distance from the origin.
Ans. $(xy' - y)^2 = 1 + (y')^2$

22. Find the differential equation of all circles in the plane.
Hint: Use $x^2 + y^2 - 2Ax - 2By + C = 0$. *Ans.* $[1 + (y')^2]y''' - 3y'(y'')^2 = 0$

CHAPTER 2

Solutions of Differential Equations

THE PROBLEM in elementary differential equations is essentially that of recovering the primitive which gave rise to the equation. In other words, the problem of solving a differential equation of order n is essentially that of finding a relation between the variables involving n independent arbitrary constants which together with the derivatives obtained from it satisfy the differential equation. For example:

Differential Equation	Primitive	
1) $\frac{d^3y}{dx^3} = 0$	$y = Ax^2 + Bx + C$	(Prob. 6, Chap. 1)
2) $\frac{d^3y}{dx^3} - 6\frac{d^2y}{dx^2} + 11\frac{dy}{dx} - 6y = 0$	$y = C_1e^{3x} + C_2e^{2x} + C_3e^{x}$	(Prob. 10, Chap. 1)
3) $y^2(\frac{dy}{dx})^2 + y^2 = r^2$	$(x-C)^2 + y^2 = r^2$	(Prob. 12, Chap. 1)

THE CONDITIONS under which we can be assured that a differential equation is solvable are given by *Existence Theorems*.

For example, a differential equation of the form $y' = g(x,y)$ for which

a) $g(x,y)$ is continuous and single valued over a region R of points (x,y),

b) $\frac{\partial g}{\partial y}$ exists and is continuous at all points in R,

admits an infinity of solutions $f(x,y,C) = 0$ (C, an arbitrary constant) such that through each point of R there passes one and only one curve of the family $f(x,y,C) = 0$. See Problem 5.

A PARTICULAR SOLUTION of a differential equation is one obtained from the primitive by assigning definite values to the arbitrary constants. For example, in 1) above $y = 0$ ($A = B = C = 0$), $y = 2x + 5$ ($A = 0$, $B = 2$, $C = 5$), and $y = x^2 + 2x + 3$ ($A = 1$, $B = 2$, $C = 3$) are particular solutions.

Geometrically, the primitive is the equation of a family of curves and a particular solution is the equation of some one of the curves. These curves are called *integral curves* of the differential equation.

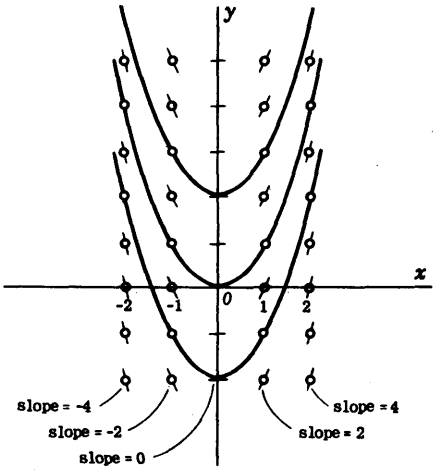
As will be seen from Problem 6, a given form of the primitive may not include all of the particular solutions. Moreover, as will be seen from Problem 7, a differential equation may have solutions which cannot be obtained from the primitive by any manipulation of the arbitrary constant as in Problem 6. Such solutions, called *singular solutions*, will be considered in Chapter 10.

The primitive of a differential equation is usually called *the general solution* of the equation. Certain authors, because of the remarks in the paragraph above, call it *a general solution* of the equation.

A DIFFERENTIAL EQUATION $\frac{dy}{dx} = g(x,y)$ associates with each point (x_0, y_0) in the region R of the above existence theorem a direction $m = \left.\frac{dy}{dx}\right|_{(x_0,y_0)} = g(x_0, y_0)$. The direction at each such point is that of the tangent to the curve of the family $f(x,y,C) = 0$, that is, the primitive, passing through the point.

The region R with the direction at each of its points indicated is called a *direction field*. In the adjoining figure, a number of points with the direction at each is shown for the equation $dy/dx = 2x$. The integral curves of the differential equation are those curves having at each of their points the direction given by the equation. In this example, the integral curves are parabolas.

Such diagrams are helpful in that they aid in clarifying the relation between a differential equation and its primitive, but since the integral curves are generally quite complex, such a diagram does not aid materially in obtaining their equations.



SOLVED PROBLEMS

1. Show by direct substitution in the differential equation and a check of the arbitrary constants that each primitive gives rise to the corresponding differential equation.

a) $y = C_1 \sin x + C_2 x$ $\qquad (1 - x\cot x)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = 0$

b) $y = C_1 e^x + C_2 x e^x + C_3 e^{-x} + 2x^2 e^x$ $\qquad \frac{d^3y}{dx^3} - \frac{d^2y}{dx^2} - \frac{dy}{dx} + y = 8e^x$

a) Substitute $y = C_1 \sin x + C_2 x$, $\frac{dy}{dx} = C_1 \cos x + C_2$, $\frac{d^2y}{dx^2} = -C_1 \sin x$ in the differential equation to obtain

$$(1 - x\cot x)(-C_1 \sin x) - x(C_1 \cos x + C_2) + (C_1 \sin x + C_2 x) =$$
$$-C_1 \sin x + C_1 x \cos x - C_1 x \cos x - C_2 x + C_1 \sin x + C_2 x = 0.$$

The order of the differential equation (2) and the number of arbitrary constants (2) agree.

b)
$$\begin{aligned} y &= C_1 e^x + C_2 x e^x + C_3 e^{-x} + 2x^2 e^x, \\ y' &= (C_1 + C_2)e^x + C_2 x e^x - C_3 e^{-x} + 2x^2 e^x + 4xe^x, \\ y'' &= (C_1 + 2C_2)e^x + C_2 x e^x + C_3 e^{-x} + 2x^2 e^x + 8xe^x + 4e^x, \\ y''' &= (C_1 + 3C_2)e^x + C_2 x e^x - C_3 e^{-x} + 2x^2 e^x + 12xe^x + 12e^x, \end{aligned}$$

and $y''' - y'' - y' + y = 8e^x$. The order of the differential equation and the number of arbitrary constants agree.

2. Show that $y = 2x + Ce^x$ is the primitive of the differential equation $\frac{dy}{dx} - y = 2(1 - x)$ and find the particular solution satisfied by $x = 0$, $y = 3$ (*i.e.*, the equation of the integral curve through (0,3)).

Substitute $y = 2x + Ce^x$ and $\dfrac{dy}{dx} = 2 + Ce^x$ in the differential equation to obtain $2 + Ce^x - (2x + Ce^x) = 2 - 2x$. When $x = 0$, $y = 3$, $3 = 2\cdot 0 + Ce^0$ and $C = 3$. The particular solution is $y = 2x + 3e^x$.

3. Show that $y = C_1e^x + C_2e^{2x} + x$ is the primitive of the differential equation $\dfrac{d^2y}{dx^2} - 3\dfrac{dy}{dx} + 2y = 2x - 3$ and find the equation of the integral curve through the points (0,0) and (1,0).

Substitute $y = C_1e^x + C_2e^{2x} + x$, $\dfrac{dy}{dx} = C_1e^x + 2C_2e^{2x} + 1$, $\dfrac{d^2y}{dx^2} = C_1e^x + 4C_2e^{2x}$ in the differential equation to obtain $C_1e^x + 4C_2e^{2x} - 3(C_1e^x + 2C_2e^{2x} + 1) + 2(C_1e^x + C_2e^{2x} + x) = 2x - 3$.

When $x = 0$, $y = 0$: $C_1 + C_2 = 0$. When $x = 1$, $y = 0$: $C_1e + C_2e^2 = -1$.

Then $C_1 = -C_2 = \dfrac{1}{e^2 - e}$ and the required equation is $y = x + \dfrac{e^x - e^{2x}}{e^2 - e}$.

4. Show that $(y - C)^2 = Cx$ is the primitive of the differential equation $4x\left(\dfrac{dy}{dx}\right)^2 + 2x\dfrac{dy}{dx} - y = 0$ and find the equations of the integral curves through the point (1,2).

Here $2(y - C)\dfrac{dy}{dx} = C$ and $\dfrac{dy}{dx} = \dfrac{C}{2(y-C)}$.

$$\text{Then}\quad 4x\frac{C^2}{4(y-C)^2} + 2x\frac{C}{2(y-C)} - y = \frac{C^2x + Cx(y-C) - y(y-C)^2}{(y-C)^2} = \frac{y[Cx - (y-C)^2]}{(y-C)^2} = 0.$$

When $x = 1$, $y = 2$: $(2 - C)^2 = C$ and $C = 1, 4$.

The equations of the integral curves through (1,2) are $(y - 1)^2 = x$ and $(y - 4)^2 = 4x$.

5. The primitive of the differential equation $\dfrac{dy}{dx} = \dfrac{y}{x}$ is $y = Cx$. Find the equation of the integral curve through *a*) (1,2) and *b*) (0,0).

a) When $x = 1$, $y = 2$: $C = 2$ and the required equation is $y = 2x$.

b) When $x = 0$, $y = 0$: C is not determined, that is, all of the integral curves pass through the origin. Note that $g(x,y) = y/x$ is not continuous at the origin and hence the existence theorem assures one and only one curve of the family $y = Cx$ through each point of the plane except the origin.

6. Differentiating $xy = C(x-1)(y-1)$ and substituting for C, we obtain the differential equation

$$x\frac{dy}{dx} + y = C\{(x-1)\frac{dy}{dx} + y - 1\} = \frac{xy}{(x-1)(y-1)}\{(x-1)\frac{dy}{dx} + y - 1\}$$

or 1) $$x(x-1)\frac{dy}{dx} + y(y-1) = 0.$$

Now both $y = 0$ and $y = 1$ are solutions of 1), since, for each, $dy/dx = 0$ and 1) is satisfied. The first is obtained from the primitive by setting $C = 0$, but the second $y = 1$ cannot be obtained by assigning a finite value to C. Similarly, 1) may be obtained from the primitive in the form $Bxy = (x-1)(y-1)$. Now the solution $y = 1$ is obtained by setting $B = 0$ while the solution $y = 0$ cannot be obtained by assigning a finite value to B. Thus, the given form of a primitive may not include all of the particular solutions of the differential equation. (Note that $x = 1$ is also a particular solution.)

7. Differentiating $y = Cx + 2C^2$, solving for $C = \frac{dy}{dx}$, and substituting in the primitive yields the differential equation

$$1)\qquad 2\left(\frac{dy}{dx}\right)^2 + x\left(\frac{dy}{dx}\right) - y = 0.$$

Since $y = -\frac{1}{8}x^2$, $\frac{dy}{dx} = -\frac{1}{4}x$ satisfies 1), $x^2 + 8y = 0$ is a solution of 1).

Now the primitive is represented by a family of straight lines and it is clear that the equation of a parabola cannot be obtained by manipulating the arbitrary constant. Such a solution is called a singular solution of the differential equation.

8. Verify and reconcile the fact that $y = C_1 \cos x + C_2 \sin x$ and $y = A\cos(x+B)$ are primitives of $\frac{d^2y}{dx^2} + y = 0$.

From $y = C_1\cos x + C_2 \sin x$, $y' = -C_1 \sin x + C_2 \cos x$ and

$$y'' = -C_1\cos x - C_2 \sin x = -y \quad\text{or}\quad \frac{d^2y}{dx^2} + y = 0.$$

From $y = A\cos(x+B)$, $y' = -A\sin(x+B)$ and $y'' = -A\cos(x+B) = -y$.

Now $y = A\cos(x+B) = A(\cos x \cos B - \sin x \sin B)$

$$= (A\cos B)\cos x + (-A\sin B)\sin x = C_1\cos x + C_2 \sin x.$$

9. Show that $\ln x^2 + \ln\frac{y^2}{x^2} = A + x$ may be written as $y^2 = Be^x$.

$$\ln x^2 + \ln\frac{y^2}{x^2} = \ln\left(x^2\frac{y^2}{x^2}\right) = \ln y^2 = A + x. \quad \text{Then } y^2 = e^{A+x} = e^A \cdot e^x = Be^x.$$

10. Show that $\text{Arc}\sin x - \text{Arc}\sin y = A$ may be written as $x\sqrt{1-y^2} - y\sqrt{1-x^2} = B$.

$\sin(\text{Arc}\sin x - \text{Arc}\sin y) = \sin A = B$.

Then $\sin(\text{Arc}\sin x)\cos(\text{Arc}\sin y) - \cos(\text{Arc}\sin x)\sin(\text{Arc}\sin y) = x\sqrt{1-y^2} - y\sqrt{1-x^2} = B$.

11. Show that $\ln(1+y) + \ln(1+x) = A$ may be written as $xy + x + y = C$.

$\ln(1+y) + \ln(1+x) = \ln(1+y)(1+x) = A$.

Then $(1+y)(1+x) = xy + x + y + 1 = e^A = B$ and $xy + x + y = B - 1 = C$.

12. Show that $\sinh y + \cosh y = Cx$ may be written as $y = \ln x + A$.

Here $\sinh y + \cosh y = \frac{1}{2}(e^y - e^{-y}) + \frac{1}{2}(e^y + e^{-y}) = e^y = Cx$.

Then $y = \ln C + \ln x = A + \ln x$.

SUPPLEMENTARY PROBLEMS

Show that each of the following expressions is a solution of the corresponding differential equation. Classify each as a particular solution or general solution (primitive).

13. $y = 2x^2$,	$xy' = 2y$.	Particular solution
14. $x^2 + y^2 = C$,	$yy' + x = 0$.	Primitive
15. $y = Cx + C^4$,	$y = xy' + (y')^4$.	Primitive
16. $(1-x)y^2 = x^3$,	$2x^3y' = y(y^2 + 3x^2)$.	Particular solution
17. $y = e^x(1+x)$,	$y'' - 2y' + y = 0$.	Particular solution
18. $y = C_1x + C_2e^x$,	$(x-1)y'' - xy' + y = 0$.	General solution
19. $y = C_1e^x + C_2e^{-x}$,	$y'' - y = 0$.	General solution
20. $y = C_1e^x + C_2e^{-x} + x - 4$,	$y'' - y = 4 - x$.	General solution
21. $y = C_1e^x + C_2e^{2x}$,	$y'' - 3y' + 2y = 0$.	General solution
22. $y = C_1e^x + C_2e^{2x} + x^2e^x$,	$y'' - 3y' + 2y = 2e^x(1-x)$.	General solution

CHAPTER 3

Equations of First Order and First Degree

A DIFFERENTIAL EQUATION of the first order and first degree may be written in the form

$$1) \qquad M(x,y)\,dx + N(x,y)\,dy = 0.$$

EXAMPLE 1. *a*) $\frac{dy}{dx} + \frac{y+x}{y-x} = 0$ may be written as $(y+x)\,dx + (y-x)\,dy = 0$ in which $M(x,y) = y+x$ and $N(x,y) = y-x$.

b) $\frac{dy}{dx} = 1 + x^2y$ may be written as $(1+x^2y)\,dx - dy = 0$ in which $M(x,y) = 1+x^2y$ and $N(x,y) = -1$.

If $M(x,y)\,dx + N(x,y)\,dy$ is the complete differential of a function $\mu(x,y)$, that is, if

$$M(x,y)\,dx + N(x,y)\,dy = d\mu(x,y).$$

1) is called exact and $\mu(x,y) = C$ is its primitive or general solution.

EXAMPLE 2. $3x^2y^2\,dx + 2x^3y\,dy = 0$ is an exact differential equation since $3x^2y^2\,dx + 2x^3y\,dy = d(x^3y^2)$. Its primitive is $x^3y^2 = C$.

If 1) is not exact but

$$\xi(x,y)\{M(x,y)\,dx + N(x,y)\,dy\} = d\mu(x,y),$$

$\xi(x,y)$ is called an integrating factor of 1) and $\mu(x,y) = C$ is its primitive.

EXAMPLE 3. $3y\,dx + 2x\,dy = 0$ is not an exact differential equation but when multiplied by $\xi(x,y) = x^2y$, we have $3x^2y^2\,dx + 2x^3y\,dy = 0$ which is exact. Hence, the primitive of $3y\,dx + 2x\,dy = 0$ is $x^3y^2 = C$. See Example 2.

If 1) is not exact and no integrating factor can be found readily, it may be possible by a change of one or both of the variables to obtain an equation for which an integrating factor can be found.

EXAMPLE 4. The transformation $x = t - y$, $dx = dt - dy$, (i.e., $x + y = t$), reduces the equation $(x+y+1)\,dx + (2x+2y+3)\,dy = 0$

to $(t+1)(dt - dy) + (2t+3)\,dy = 0$

or $(t+1)\,dt + (t+2)\,dy = 0.$

By means of the integrating factor $\frac{1}{t+2}$ the equation takes the form

$$dy + \frac{t+1}{t+2}dt = dy + dt - \frac{1}{t+2}dt = 0.$$

Then $y + t - \ln(t+2) = C$

and, since $t = x+y$, $2y + x - \ln(x+y+2) = C$.

Note. The transformation $x+y+1 = t$ or $2x+2y+3 = 2s$ is also suggested by the form of the equation.

A DIFFERENTIAL EQUATION for which an integrating factor is found readily has the form

2) $$f_1(x)\cdot g_2(y)\,dx + f_2(x)\cdot g_1(y)\,dy = 0.$$

By means of the integrating factor $\dfrac{1}{f_2(x)\cdot g_2(y)}$, 2) is reduced to

2') $$\frac{f_1(x)}{f_2(x)}\,dx + \frac{g_1(y)}{g_2(y)}\,dy = 0$$

whose primitive is

$$\int \frac{f_1(x)}{f_2(x)}\,dx + \int \frac{g_1(y)}{g_2(y)}\,dy = C.$$

Equation 2) is typed as *Variables Separable* and in 2') the variables are separated.

EXAMPLE 5. When the differential equation

$$(3x^2y - xy)\,dx + (2x^3y^2 + x^3y^4)\,dy = 0$$

is put in the form $\quad y(3x^2 - x)\,dx + x^3(2y^2 + y^4)\,dy = 0$

it is seen to be of the type Variables Separable. The integrating factor $\dfrac{1}{yx^3}$ reduces it to $(\frac{3}{x} - \frac{1}{x^2})\,dx + (2y + y^3)\,dy = 0$ in which the variables are separated. Integrating, we obtain the primitive

$$3\ln x + \frac{1}{x} + y^2 + \frac{1}{4}y^4 = C.$$

IF EQUATION 1) admits a solution $f(x,y,C) = 0$, where C is an arbitrary constant, there exist infinitely many integrating factors $\xi(x,y)$ such that

$$\xi(x,y)\{M(x,y)\,dx + N(x,y)\,dy\} = 0$$

is exact. Also, there exist transformations of the variables which carry 1) into the type Variables Separable. However, no general rule can be stated here for finding either an integrating factor or a transformation. Thus we are limited to solving certain types of differential equations of the first order and first degree, i.e., those for which rules may be laid down for determining either an integrating factor or an effective transformation.

Equations of the type Variables Separable, together with equations which can be reduced to this type by a transformation of the variables are considered in Chapter 4.

Exact differential equations and other types reducible to exact equations by means of integrating factors are treated in Chapter 5.

The linear equation of order one

3) $$\frac{dy}{dx} + P(x)\cdot y = Q(x)$$

and equations reducible to the form 3) by means of transformations are considered in Chapter 6.

These groupings are a matter of convenience. A given equation may fall into more than one group.

Example 6. The equation $x\,dy - y\,dx = 0$ may be placed in any one of the groups since

a) by means of the integrating factor $1/xy$ the variables are separated; thus, $dy/y - dx/x = 0$ so that $\ln y - \ln x = \ln C$ or $y/x = C$.

b) by means of the integrating factor $1/x^2$ or $1/y^2$ the equation is made exact; thus, $\dfrac{x\,dy - y\,dx}{x^2} = 0$ and $\dfrac{y}{x} = C$ or $\dfrac{x\,dy - y\,dx}{y^2} = 0$ and $-\dfrac{x}{y} = C_1$, $\dfrac{y}{x} = -\dfrac{1}{C_1} = C$.

c) when written as $\dfrac{dy}{dx} - \dfrac{1}{x}y = 0$, it is a linear equation of order one.

Attention has been called to the fact that the form of the primitive is not unique. Thus, the primitive in Example 6 might be given as

a) $\ln y - \ln x = \ln C$, *b*) $y/x = C$, *c*) $y = Cx$, *d*) $x/y = K$, etc.

It is usual to accept any one of these forms with the understanding, already noted, that thereby certain particular solutions may be lost. There is an additional difficulty!

Example 7. It is clear that $y=0$ is a particular solution of $dy/dx = y$ or $dy - y\,dx = 0$. When $y \neq 0$, we may write $dy/y - dx = 0$ and obtain $\ln y - x = \ln C$ with $C \neq 0$; in turn, this may be written as $y = Ce^x$, $C \neq 0$. Thus, to include all solutions, we should write $y = 0$; $y = Ce^x$, $C \neq 0$. But note that $y = Ce^x$, free of the restrictions imposed on y and C, includes *all* solutions.

This situation will arise repeatedly as we proceed but, as is customary, we shall refrain from pointing out the restrictions; that is, we shall write the primitive as $y = Ce^x$, with C completely arbitrary. In defense, we offer the following observation. Let us multiply the given equation by e^{-x} to obtain $e^{-x}dy - ye^{-x}dx = 0$ from which, by integration, we get $e^{-x}y = C$ or $y = Ce^x$. In this procedure, it has not been necessary to impose any restriction on y or C.

CHAPTER 4

Equations of First Order and First Degree

VARIABLES SEPARABLE AND REDUCTION TO VARIABLES SEPARABLE

VARIABLES SEPARABLE. The variables of the equation $M(x,y)\,dx + N(x,y)\,dy = 0$ are separable if the equation can be written in the form

1) $$f_1(x)\cdot g_2(y)\,dx + f_2(x)\cdot g_1(y)\,dy = 0.$$

The integrating factor $\dfrac{1}{f_2(x)\cdot g_2(y)}$, found by inspection, reduces 1) to the form

$$\frac{f_1(x)}{f_2(x)}\,dx + \frac{g_1(y)}{g_2(y)}\,dy = 0$$

from which the primitive may be obtained by integration.

For example, $(x-1)^2 y\,dx + x^2(y+1)\,dy = 0$ is of the form 1). The integrating factor $\dfrac{1}{x^2 y}$ reduces the equation to $\dfrac{(x-1)^2}{x^2}\,dx + \dfrac{(y+1)}{y}\,dy = 0$ in which the variables are separated. See Problems 1-5.

HOMOGENEOUS EQUATIONS. A function $f(x,y)$ is called homogeneous of degree n if

$$f(\lambda x, \lambda y) = \lambda^n f(x,y).$$

For example:

a) $f(x,y) = x^4 - x^3 y$ is homogeneous of degree 4 since

$$f(\lambda x, \lambda y) = (\lambda x)^4 - (\lambda x)^3(\lambda y) = \lambda^4(x^4 - x^3 y) = \lambda^4 f(x,y).$$

b) $f(x,y) = e^{y/x} + \tan\frac{y}{x}$ is homogeneous of degree 0 since

$$f(\lambda x, \lambda y) = e^{\lambda y/\lambda x} + \tan\frac{\lambda y}{\lambda x} = e^{y/x} + \tan\frac{y}{x} = \lambda^0 f(x,y).$$

c) $f(x,y) = x^2 + \sin x\cos y$ is not homogeneous since

$$f(\lambda x, \lambda y) = \lambda^2 x^2 + \sin(\lambda x)\cos(\lambda y) \neq \lambda^n f(x,y).$$

The differential equation $M(x,y)\,dx + N(x,y)\,dy = 0$ is called homogeneous if $M(x,y)$ and $N(x,y)$ are homogeneous and of the same degree. For example, $x\ln\frac{y}{x}\,dx + \frac{y^2}{x}\arcsin\frac{y}{x}\,dy = 0$ is homogeneous of degree 1, but neither $(x^2+y^2)\,dx - (xy^2 - y^3)\,dy = 0$ nor $(x+y^2)\,dx + (x-y)\,dy = 0$ is a homogeneous equation.

The transformation $y = vx, \quad dy = v\,dx + x\,dv$ will reduce any homogeneous equation to the form

$$P(x,v)\,dx + Q(x,v)\,dv = 0$$

in which the variables are separable. After integrating, v is replaced by y/x to recover the original variables. See Problems 6-11.

EQUATIONS IN WHICH $M(x,y)$ AND $N(x,y)$ ARE LINEAR BUT NOT HOMOGENEOUS.

a) The equation $(a_1x + b_1y + c_1)dx + (a_2x + b_2y + c_2)dy = 0$, $(a_1b_2 - a_2b_1 = 0)$, is reduced by the transformation

$$a_1x + b_1y = t, \qquad dy = \frac{dt - a_1dx}{b_1}$$

to the form

$$P(x,t)\,dx + Q(x,t)\,dt = 0$$

in which the variables are separable. See Problem 12.

b) The equation $(a_1x + b_1y + c_1)dx + (a_2x + b_2y + c_2)dy = 0$, $(a_1b_2 - a_2b_1 \neq 0)$, is reduced to the homogeneous form

$$(a_1x' + b_1y')\,dx' + (a_2x' + b_2y')\,dy' = 0$$

by the transformation $x = x' + h, \quad y = y' + k$

in which $x = h$, $y = k$ are the solutions of the equations

$a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$. See Problems 13-14.

EQUATIONS OF THE FORM $y\cdot f(xy)\,dx + x\cdot g(xy)\,dy = 0$. The transformation

$$xy = z, \qquad y = \frac{z}{x}, \qquad dy = \frac{x\,dz - z\,dx}{x^2}$$

reduces an equation of this form to the form

$$P(x,z)\,dx + Q(x,z)\,dz = 0$$

in which the variables are separable. See Problems 15-17.

OTHER SUBSTITUTIONS. Equations, not of the types discussed above, may be reduced to a form in which the variables are separable by means of a properly chosen transformation. No general rule of procedure can be given; in each case the form of the equation suggests the transformation. See Problems 18-22.

SOLVED PROBLEMS

VARIABLES SEPARABLE.

1. Solve $x^3dx + (y+1)^2dy = 0$.

The variables are separated. Hence, integrating term by term,

$$\frac{x^4}{4} + \frac{(y+1)^3}{3} = C_1 \quad \text{or} \quad 3x^4 + 4(y+1)^3 = C.$$

2. Solve $x^2(y+1)dx + y^2(x-1)dy = 0$.

The integrating factor $\dfrac{1}{(y+1)(x-1)}$ reduces the equation to $\dfrac{x^2}{x-1}dx + \dfrac{y^2}{y+1}dy = 0$.

Then, integrating $(x+1+\dfrac{1}{x-1})dx + (y-1+\dfrac{1}{y+1})dy = 0$,

$$\tfrac{1}{2}x^2 + x + \ln(x-1) + \tfrac{1}{2}y^2 - y + \ln(y+1) = C_2,$$

$$x^2 + y^2 + 2x - 2y + 2\ln(x-1)(y+1) = C_1,$$

and
$$(x+1)^2 + (y-1)^2 + 2\ln(x-1)(y+1) = C.$$

3. Solve $4x\,dy - y\,dx = x^2dy$ or $y\,dx + (x^2-4x)dy = 0$.

The integrating factor $\dfrac{1}{y(x^2-4x)}$ reduces the equation to $\dfrac{dx}{x(x-4)} + \dfrac{dy}{y} = 0$ in which the variables are separated.

The latter equation may be written as $\dfrac{\frac{1}{4}\,dx}{x-4} - \dfrac{\frac{1}{4}\,dx}{x} + \dfrac{dy}{y} = 0$ or $\dfrac{dx}{x-4} - \dfrac{dx}{x} + 4\dfrac{dy}{y} = 0.$

Integrating, $\ln(x-4) - \ln x + 4\ln y = \ln C$ or $(x-4)y^4 = Cx$.

4. Solve $\dfrac{dy}{dx} = \dfrac{4y}{x(y-3)}$ or $x(y-3)dy = 4y\,dx$.

The integrating factor $\dfrac{1}{xy}$ reduces the equation to $\dfrac{y-3}{y}dy = \dfrac{4}{x}dx$.

Integrating, $y - 3\ln y = 4\ln x + \ln C_1$ or $y = \ln(C_1x^4y^3)$.

This may be written as $C_1x^4y^3 = e^y$ or $x^4y^3 = Ce^y$.

5. Find the particular solution of $(1+x^3)dy - x^2y\,dx = 0$ satisfying the initial conditions $x = 1$, $y = 2$.

First find the primitive, using the integrating factor $\dfrac{1}{y(1+x^3)}$.

Then $\dfrac{dy}{y} - \dfrac{x^2}{1+x^3}dx = 0$, $\ln y - \dfrac{1}{3}\ln(1+x^3) = C_1$, $3\ln y = \ln(1+x^3) + \ln C$, $y^3 = C(1+x^3)$.

When $x = 1$, $y = 2$: $2^3 = C(1+1)$, $C = 4$, and the required particular solution is $y^3 = 4(1+x^3)$.

HOMOGENEOUS EQUATIONS.

✓ **6.** When $Mdx + Ndy = 0$ is homogeneous, show that the transformation $y = vx$ will separate the variables.

When $Mdx + Ndy = 0$ is homogeneous of degree n, we may write

$$Mdx + N\,dy = x^n\{M_1(\tfrac{y}{x})dx + N_1(\tfrac{y}{x})dy\} = 0 \quad \text{whence} \quad M_1(\tfrac{y}{x})dx + N_1(\tfrac{y}{x})dy = 0.$$

The transformation $y = vx$, $dy = v\,dx + x\,dv$ reduces this to

$$M_1(v)\,dx + N_1(v)\{v\,dx + x\,dv\} = 0 \quad \text{or} \quad \{M_1(v) + vN_1(v)\}dx + xN_1(v)\,dv = 0$$

or, finally, $\dfrac{dx}{x} + \dfrac{N_1(v)\,dv}{M_1(v) + vN_1(v)} = 0$ in which the variables are separated.

7. Solve $(x^3+y^3)dx - 3xy^2dy = 0.$

The equation is homogeneous of degree 3. We use the transformation $y = vx$, $dy = v\,dx + x\,dv$ to obtain

$$1)\quad x^3\{(1+v^3)dx - 3v^2(v\,dx + x\,dv)\} = 0 \qquad \text{or} \qquad (1-2v^3)dx - 3v^2x\,dv = 0$$

in which the variables are separable.

Upon separating the variables, using the integrating factor $\dfrac{1}{x(1-2v^3)}$, $\dfrac{dx}{x} - \dfrac{3v^2\,dv}{1-2v^3} = 0,$

and $\ln x + \frac{1}{2}\ln(1-2v^3) = C_1,\quad 2\ln x + \ln(1-2v^3) = \ln C,\quad$ or $\quad x^2(1-2v^3) = C.$

Since $v = y/x$, the primitive is $x^2(1-2y^3/x^3) = C$ or $x^3 - 2y^3 = Cx.$

Note that the equation is of degree 3 and that after the transformation x^3 is a factor of the left member of 1). This factor may be removed when making the transformation.

8. Solve $x\,dy - y\,dx - \sqrt{x^2-y^2}\,dx = 0.$

The equation is homogeneous of degree 1. Using the transformation $y=vx$, $dy = v\,dx + x\,dv$ and dividing by x, we have

$$v\,dx + x\,dv - v\,dx - \sqrt{1-v^2}\,dx = 0 \qquad \text{or} \qquad x\,dv - \sqrt{1-v^2}\,dx = 0.$$

When the variables are separated, using the integrating factor $\dfrac{1}{x\sqrt{1-v^2}}$, $\dfrac{dv}{\sqrt{1-v^2}} - \dfrac{dx}{x} = 0.$

Then $\arcsin v - \ln x = \ln C$ or $\arcsin v = \ln(Cx)$, and returning to the original variables, using $v = y/x$, $\arcsin\dfrac{y}{x} = \ln(Cx)$ or $Cx = e^{\arcsin y/x}$.

9. Solve $(2x\sinh\frac{y}{x} + 3y\cosh\frac{y}{x})dx - 3x\cosh\frac{y}{x}\,dy = 0.$

The equation is homogeneous of degree 1. Using the standard transformation and dividing by x, we have

$$2\sinh v\,dx - 3x\cosh v\,dv = 0.$$

Then, separating the variables, $2\dfrac{dx}{x} - 3\dfrac{\cosh v}{\sinh v}\,dv = 0.$

Integrating, $2\ln x - 3\ln\sinh v = \ln C,\quad x^2 = C\sinh^3 v,\quad$ and $\quad x^2 = C\sinh^3\dfrac{y}{x}.$

10. Solve $(2x+3y)dx + (y-x)dy = 0.$

The equation is homogeneous of degree 1. The standard transformation reduces it to

$$(2+3v)dx + (v-1)(v\,dx + x\,dv) = 0 \qquad \text{or} \qquad (v^2+2v+2)dx + x(v-1)dv = 0.$$

Separating the variables, $\dfrac{dx}{x} + \dfrac{v-1}{v^2+2v+2}\,dv = \dfrac{dx}{x} + \frac{1}{2}\dfrac{2v+2}{v^2+2v+2}\,dv - \dfrac{2\,dv}{(v+1)^2+1} = 0.$

Integrating, $\ln x + \frac{1}{2}\ln(v^2+2v+2) - 2\arctan(v+1) = C_1,$

$\ln x^2(v^2+2v+2) - 4\arctan(v+1) = C,\quad$ and $\quad \ln(y^2+2xy+2x^2) - 4\arctan\dfrac{x+y}{x} = C.$

11. Solve $(1+2e^{x/y})dx + 2e^{x/y}(1-\frac{x}{y})dy = 0.$

The equation is homogeneous of degree 0. The appearance of x/y throughout the equation suggests the use of the transformation $x = vy$, $dx = v\,dy + y\,dv$.

Then $(1+2e^{v})(v\,dy + y\,dv) + 2e^{v}(1-v)dy = 0$, $(v+2e^{v})dy + y(1+2e^{v})dv = 0$,

and
$$\frac{dy}{y} + \frac{1+2e^{v}}{v+2e^{v}}\,dv = 0.$$

Integrating and replacing v by x/y, $\ln y + \ln(v+2e^{v}) = \ln C$ and $x + 2ye^{x/y} = C$.

LINEAR BUT NOT HOMOGENEOUS.

12. Solve $(x+y)dx + (3x+3y-4)dy = 0$.

The expressions $(x+y)$ and $(3x+3y)$ suggest the transformation $x+y = t$.

We use $y = t-x$, $dy = dt-dx$ to obtain $t\,dx + (3t-4)(dt-dx) = 0$

or $(4-2t)dx + (3t-4)dt = 0$

in which the variables are separable.

Then $2dx + \dfrac{3t-4}{2-t}\,dt = 2\,dx - 3\,dt + \dfrac{2}{2-t}\,dt = 0$.

Integrating and replacing t by $x+y$, we have

$2x-3t-2\ln(2-t) = C_1$, $2x-3(x+y)-2\ln(2-x-y) = C_1$, and $x+3y+2\ln(2-x-y) = C$.

13. Solve $(2x-5y+3)dx - (2x+4y-6)dy = 0$.

First solve $2x-5y+3 = 0$, $2x+4y-6 = 0$ simultaneously to obtain $x = h = 1$, $y = k = 1$.

The transformation
$$x = x'+h = x'+1,\quad dx = dx'$$
$$y = y'+k = y'+1,\quad dy = dy'$$

reduces the given equation to $(2x'-5y')dx' - (2x'+4y')dy' = 0$

which is homogeneous of degree 1. (Note that this latter equation can be written down without carrying out the details of the transformation.)

Using the transformation $y' = vx'$, $dy' = v\,dx' + x'\,dv$,

we obtain $(2-5v)dx' - (2+4v)(v\,dx' + x'\,dv) = 0$, $(2-7v-4v^2)dx' - x'(2+4v)dv = 0$,

and finally
$$\frac{dx'}{x'} + \frac{4}{3}\frac{dv}{4v-1} + \frac{2}{3}\frac{dv}{v+2} = 0.$$

Integrating, $\ln x' + \dfrac{1}{3}\ln(4v-1) + \dfrac{2}{3}\ln(v+2) = \ln C_1$ or $x'^3(4v-1)(v+2)^2 = C$.

Replacing v by y'/x', $(4y'-x')(y'+2x')^2 = C$,

and replacing x' by $x-1$ and y' by $y-1$, we obtain the primitive $(4y-x-3)(y+2x-3)^2 = C$.

14. Solve $(x-y-1)dx + (4y+x-1)dy = 0$.

Solving $x-y-1=0$, $4y+x-1=0$ simultaneously, we obtain $x=h=1$, $y=k=0$.

The transformation
$$x = x'+h = x'+1,\quad dx = dx'$$
$$y = y'+k = y'\quad,\quad dy = dy'$$

reduces the given equation to $(x'-y')dx' + (4y'+x')dy' = 0$ which is homogeneous of degree 1. (Note that this transformation $x-1 = x'$, $y=y'$ could have been obtained by inspection, that is, by examining the terms $(x-y-1)$ and $(4y+x-1)$.)

Using the transformation $y' = vx', \quad dy' = v\,dx' + x'\,dv$

we obtain $$(1-v)dx' + (4v+1)(v\,dx' + x'\,dv) = 0.$$

Then $$\frac{dx'}{x'} + \frac{4v+1}{4v^2+1}\,dv = \frac{dx'}{x'} + \frac{1}{2}\,\frac{8v}{4v^2+1}\,dv + \frac{dv}{4v^2+1} = 0,$$

$$\ln x' + \tfrac{1}{2}\ln(4v^2+1) + \tfrac{1}{2}\arctan 2v = C_1, \qquad \ln x'^2(4v^2+1) + \arctan 2v = C,$$

$$\ln(4y'^2 + x'^2) + \arctan\frac{2y'}{x'} = C, \quad \text{and} \quad \ln[4y^2 + (x-1)^2] + \arctan\frac{2y}{x-1} = C.$$

FORM $y\,f(xy)dx + x\,g(xy)dy = 0$.

15. Solve $y(xy+1)dx + x(1+xy+x^2y^2)dy = 0.$

The transformation $xy = v, \quad y = v/x, \quad dy = \dfrac{x\,dv - v\,dx}{x^2}$

reduces the equation to $$\frac{v}{x}(v+1)dx + x(1+v+v^2)\,\frac{x\,dv - v\,dx}{x^2} = 0$$

which, after clearing of fractions and rearranging, becomes $v^3\,dx - x(1+v+v^2)dv = 0.$

Separating the variables, we have $$\frac{dx}{x} - \frac{dv}{v^3} - \frac{dv}{v^2} - \frac{dv}{v} = 0.$$

Then $$\ln x + \frac{1}{2v^2} + \frac{1}{v} - \ln v = C_1, \qquad 2v^2\ln\left(\frac{v}{x}\right) - 2v - 1 = Cv^2,$$

and $$2x^2y^2\ln y - 2xy - 1 = Cx^2y^2.$$

16. Solve $(y - xy^2)dx - (x + x^2y)dy = 0$ or $y(1-xy)dx - x(1+xy)dy = 0.$

The transformation $xy = v, \quad y = v/x, \quad dy = \dfrac{x\,dv - v\,dx}{x^2}$ reduces the equation to

$$\frac{v}{x}(1-v)dx - x(1+v)\,\frac{x\,dv - v\,dx}{x^2} = 0 \quad \text{or} \quad 2v\,dx - x(1+v)dv = 0.$$

Then $$2\,\frac{dx}{x} - \frac{1+v}{v}\,dv = 0, \quad 2\ln x - \ln v - v = \ln C, \quad \frac{x^2}{v} = Ce^v, \quad \text{and} \quad x = Cye^{xy}.$$

17. Solve $(1 - xy + x^2y^2)dx + (x^3y - x^2)dy = 0$ or $y(1 - xy + x^2y^2)dx + x(x^2y^2 - xy)dy = 0.$

The transformation $xy = v, \quad y = v/x, \quad dy = \dfrac{x\,dv - v\,dx}{x^2}$ reduces the equation to

$$\frac{v}{x}(1 - v + v^2)dx + x(v^2 - v)\,\frac{x\,dv - v\,dx}{x^2} = 0 \quad \text{or} \quad v\,dx + x(v^2 - v)dv = 0.$$

Then $$\frac{dx}{x} + (v-1)dv = 0, \quad \ln x + \tfrac{1}{2}v^2 - v = C, \quad \text{and} \quad \ln x = xy - \tfrac{1}{2}x^2y^2 + C.$$

MISCELLANEOUS SUBSTITUTIONS.

18. Solve $\dfrac{dy}{dx} = (y-4x)^2$ or $dy = (y-4x)^2\,dx$.

The suggested transformation $y-4x=v,\ dy = 4dx + dv$ reduces the equation to

$$4\,dx + dv = v^2 dx \qquad \text{or} \qquad dx - \frac{dv}{v^2-4} = 0.$$

Then $x + \frac{1}{4}\ln\dfrac{v+2}{v-2} = C_1$, $\ln\dfrac{v+2}{v-2} = \ln C - 4x$, $\dfrac{v+2}{v-2} = Ce^{-4x}$, and $\dfrac{y-4x+2}{y-4x-2} = Ce^{-4x}$.

19. Solve $\tan^2(x+y)dx - dy = 0$.

The suggested transformation $x+y = v,\ dy = dv - dx$ reduces the equation to

$$\tan^2 v\ dx - (dv - dx) = 0, \qquad dx - \frac{dv}{1+\tan^2 v} = 0, \qquad \text{or} \qquad dx - \cos^2 v\ dv = 0.$$

Integrating, $x - \frac{1}{2}v - \frac{1}{4}\sin 2v = C_1$ and $2(x-y) = C + \sin 2(x+y)$.

20. Solve $(2+2x^2y^{1/2})y\ dx + (x^2y^{1/2}+2)x\ dy = 0$.

The suggested transformation $x^2y^{1/2} = v,\ y = \dfrac{v^2}{x^4},\ dy = \dfrac{2v}{x^4}dv - \dfrac{4v^2}{x^5}dx$ reduces the equation to

$$(2+2v)\frac{v^2}{x^4}\,dx + x(v+2)\left(\frac{2v}{x^4}\,dv - \frac{4v^2}{x^5}\,dx\right) = 0 \qquad \text{or} \qquad v(3+v)dx - x(v+2)dv = 0.$$

Then $\dfrac{dx}{x} - \dfrac{2}{3}\dfrac{dv}{v} - \dfrac{1}{3}\dfrac{dv}{v+3} = 0$, $3\ln x - 2\ln v - \ln(v+3) = \ln C_1$, $x^3 = C_1v^2(v+3)$,

and $1 = C_1xy(x^2y^{1/2}+3)$ or $xy(x^2y^{1/2}+3) = C$.

21. Solve $(2x^2+3y^2-7)x\ dx - (3x^2+2y^2-8)y\ dy = 0$.

The suggested transformation $x^2 = u,\ y^2 = v$ reduces the equation to

$$(2u+3v-7)du - (3u+2v-8)dv = 0$$

which is linear but not homogeneous.

The transformation $u = s+2,\ v = t+1$ yields the homogeneous equation $(2s+3t)ds - (3s+2t)dt = 0$, and the transformation $s = rt,\ ds = r\,dt + t\,dr$ yields $2(r^2-1)dt + (2r+3)t\ dr = 0$.

Separating the variables, we have $2\dfrac{dt}{t} + \dfrac{2r+3}{r^2-1}dr = 2\dfrac{dt}{t} - \dfrac{1}{2}\dfrac{dr}{r+1} + \dfrac{5}{2}\dfrac{dr}{r-1} = 0$.

Then $4\ln t - \ln(r+1) + 5\ln(r-1) = \ln C$,

$$\frac{t^4(r-1)^5}{r+1} = \frac{(s-t)^5}{s+t} = \frac{(u-v-1)^5}{u+v-3} = \frac{(x^2-y^2-1)^5}{x^2+y^2-3} = C, \quad \text{and} \quad (x^2-y^2-1)^5 = C(x^2+y^2-3).$$

22. Solve $x^2(x\,dx + y\,dy) + y(x\,dy - y\,dx) = 0$.

Here $x\,dx + y\,dy = \frac{1}{2}d(x^2+y^2)$ and $x\,dy - y\,dx = x^2d(y/x)$ suggest $x^2+y^2 = \rho^2$, $y/x = \tan\theta$, or $x = \rho\cos\theta$, $y = \rho\sin\theta$, $dx = -\rho\sin\theta\,d\theta + \cos\theta\,d\rho$, $dy = \rho\cos\theta\,d\theta + \sin\theta\,d\rho$.

The given equation takes the form $\rho^2\cos^2\theta\,(\rho\,d\rho) + \rho\sin\theta\,(\rho^2\,d\theta) = 0$
or $d\rho + \tan\theta\sec\theta\,d\theta = 0$.

Then $\rho + \sec\theta = C_1$, $\sqrt{x^2+y^2}\left(\dfrac{x+1}{x}\right) = C_1$, and $(x^2+y^2)(x+1)^2 = Cx^2$.

SUPPLEMENTARY PROBLEMS

23. Determine whether or not each of the following functions is homogeneous and, when homogeneous, state the degree.

a) $x^2 - xy$,	homo. of degree two.
b) $\dfrac{xy}{x+y^2}$,	not homo.
c) $\dfrac{xy}{x^2+y^2}$;	homo. of degree zero.
d) $x + y\cos\dfrac{y}{x}$,	homo. of degree one.
e) $\arcsin xy$,	not homo.
f) $xe^{y/x} + ye^{x/y}$,	homo. of degree one.
g) $\ln x - \ln y$ or $\ln\dfrac{x}{y}$,	homo. of degree zero.
h) $\sqrt{x^2 + 2xy + 3y^2}$,	homo. of degree one.
i) $x\sin y + y\sin x$,	not homo.

Classify each of the equations below in one or more of the following categories:

(1) Variables separable
(2) Homogeneous equations
(3) Equations in which $M(x,y)$ and $N(x,y)$ are linear but not homogeneous
(4) Equations of the form $y\,f(xy)dx + x\,g(xy)dy = 0$
(5) None of the above apply.

24. $4y\,dx + x\,dy = 0$ *Ans.* (1); (2), of degree one

25. $(1+2y)dx + (4-x^2)dy = 0$ (1)

26. $y^2\,dx - x^2\,dy = 0$ (1); (2), of degree two

27. $(1+y)dx - (1+x)dy = 0$ (1); (3)

28. $(xy^2+y)dx + (x^2y-x)dy = 0$ (4)

29. $(x\sin\frac{y}{x} - y\cos\frac{y}{x})dx + x\cos\frac{y}{x}\,dy = 0$ (2), of degree one

30. $y^2(x^2+2)dx + (x^3+y^3)(y\,dx - x\,dy) = 0$ (5)

31. $y\sqrt{x^2+y^2}\,dx - x(x+\sqrt{x^2+y^2})dy = 0$ (2), of degree two

32. $(x+y+1)dx + (2x+2y+1)dy = 0$ (3)

33. Solve each of the above equations (Problems 24-32) which fall in categories (1)-(4).

Ans. 24. $x^4y = C$
25. $(1+2y)^2 = C\,\dfrac{2-x}{2+x}$
26. $y = x + Cxy$
27. $(1+y) = C(1+x)$
28. $y = Cxe^{xy}$
29. $x\sin\frac{y}{x} = C$
31. $Cx - \sqrt{x^2+y^2} = x\ln(\sqrt{x^2+y^2} - x)$
32. $x + 2y + \ln(x+y) = C$

Solve each of the following equations.

34. $(1+2y)dx - (4-x)dy = 0$ *Ans.* $(x-4)^2(1+2y) = C$

35. $xy\,dx + (1+x^2)dy = 0$ *Ans.* $y^2(1+x^2) = C$

36. $\cot\theta\,d\rho + \rho\,d\theta = 0$ *Ans.* $\rho = C\cos\theta$

37. $(x+2y)dx + (2x+3y)dy = 0$ *Ans.* $x^2 + 4xy + 3y^2 = C$

38. $2x\,dy - 2y\,dx = \sqrt{x^2+4y^2}\,dx$ *Ans.* $1 + 4Cy - C^2x^2 = 0$

39. $(3y-7x+7)dx + (7y-3x+3)dy = 0$ *Ans.* $(y-x+1)^2(y+x-1)^5 = C$

40. $xy\,dy = (y+1)(1-x)dx$ *Ans.* $y+x = \ln Cx(y+1)$

41. $(y^2-x^2)dx + xy\,dy = 0$ *Ans.* $2x^2y^2 = x^4 + C$

42. $y(1+2xy)dx + x(1-xy)dy = 0$ *Ans.* $y = Cx^2e^{-1/xy}$

43. $dx + (1-x^2)\cot y\,dy = 0$ *Ans.* $\sin^2 y = C\dfrac{1-x}{1+x}$

44. $(x^3+y^3)dx + 3xy^2dy = 0$ *Ans.* $x^4 + 4xy^3 = C$

45. $(3x+2y+1)dx - (3x+2y-1)dy = 0$ *Ans.* $\ln(15x+10y-1) + \frac{5}{2}(x-y) = C$

In each of the following, find the particular solution indicated.

46. $x\,dy + 2y\,dx = 0$; when $x = 2$, $y = 1$. *Ans.* $x^2y = 4$

47. $(x^2+y^2)dx + xy\,dy = 0$; when $x = 1$, $y = -1$. *Ans.* $x^4 + 2x^2y^2 = 3$

48. $\cos y\,dx + (1+e^{-x})\sin y\,dy = 0$; when $x = 0$, $y = \pi/4$. *Ans.* $(1+e^x)\sec y = 2\sqrt{2}$

49. $(y^2+xy)dx - x^2dy = 0$; when $x = 1$, $y = 1$. *Ans.* $x = e^{1-x/y}$

50. Solve the equation of Problem 30 using the substitution $y = vx$.

Ans. $x^2y\ln x - y + x^3 - \frac{1}{2}y^3 = Cx^2y$

51. Solve $y' = -2(2x+3y)^2$ using the substitution $z = 2x+3y$.

Ans. $\dfrac{1+\sqrt{3}(2x+3y)}{1-\sqrt{3}(2x+3y)} = Ce^{4\sqrt{3}x}$

52. Solve $(x - 2\sin y + 3)dx + (2x - 4\sin y - 3)\cos y\,dy = 0$ using the substitution $\sin y = z$.

Ans. $8\sin y + 4x + 9\ln(4x - 8\sin y + 3) = C$